Black holes with a negative cosmological constant

Piotr T. Chruściel

University of Vienna

Wojanów, September 2018

Black holes with a negative cosmological constant (but not only)

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Registration for CERS9 open now

https://www.univie.ac.at/cers/cers9/

9th Central European Relativity Seminar

February 14-16, 2019, Kraków

The 9th *Central European Relativity Seminar* will be held in Kraków from Thursday, February 14 to Saturday, February 16, 2019. The meeting will begin on Thursday at 2pm and end on Saturday mid-day.

The Kraków meeting will be the ninth seminar of a series initiated at the Erwin Schroedinger Institute in Vienna in 2011. Here is the list, with links, to all previous meetings.

This series of seminars is designed to provide a forum for younger researchers to present their work, and to expand their research horizons, in all topics of research in general relativity. While the main geographical basin of attraction is Austria, the Czech Republic, Hungary, Poland and Germany, we welcome researchers from all countries.

See you in Kraków in February!

Organizers:

- Lars Andersson (Golm)
- Robert Beig (Vienna)
- Piotr Bizoń (Kraków)
- Piotr T. Chruściel (Vienna)
- Helmut Friedrich (Golm)
- Maciej Maliborski (Vienna)

1. Some unresolved annoyances in a mathematically rigorous treatment of black holes, $\Lambda = 0$.
2. Recent progress, for $\Lambda \leq 0$, including a construction of stationary black holes, solutions of Einstein-matter equations (possibly vacuum).
3. Recent progress for $\Lambda > 0$.

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- Long time goal: exhaustively describe the global properties of solutions of the equations
- First step: find all *regular* time-independent solutions *(in GR, this means both strictly stationary solutions and black hole solutions)*
- Next step: isolate the *stable* ones
- Next step: explore the remainder of the iceberg (beyond event horizons: chaos? mixmaster? curvature blowup? spikes? extendable singularities? else?...)

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Some mathematical issues in any field theory:

Mathematical questions in all geometric theories of gravitation

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Static vs. stationary

Time-independent can be static or stationary;

- *static:* stationarity plus *time-reversal isometry*
- *Regular*, static, black hole *exteriors* $(\mathcal{M}, g)$ take the form $\mathcal{M} = \mathbb{R} \times \Sigma$,

$g = -V^2 dt^2 + \gamma$, and the Riemannian metric γ satisfies

$$\textit{in vacuum}: \quad V\mathrm{Ricci}(\gamma) = \mathrm{Hess}\, V \ , \ \Delta V = 0 \ ,$$

with $\partial\Sigma = \{V = 0\}$. $\partial\Sigma$ = non-degenerate horizons; asymptotically cylindrical ends = degenerate horizons

Static vs. stationary and degenerate (also known as extreme) vs. non-degenerate

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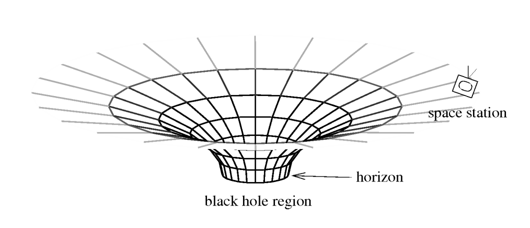

Embedding a *non-degenerate* space-geometry in higher dimension

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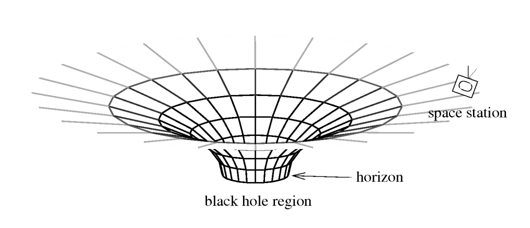

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$\Lambda = 0$: "Black Holes have No Hair"

The **analytic, nondegenerate,connected** classification in space-time dimension **four**; contributions by Israel, Hawking, Carter, Robinson, Bunting, Mazur, PTC-Costa Lopes, PTC-Sudarsky-Wald,

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Stationary, electro-vacuum, analytic, non-degenerate, connected, regular black hole = **Kerr-Newman** in the exterior region	Static, electro-vacuum, regular black hole = **MP or RN** in the exterior region

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- Consider a nearly-stationary black-hole configuration, one *expects* that it will evolve to a stationary solution
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- Similar problem with the non-degeneracy condition

Analyticity?

Some progress: Alexakis, Ionescu, Klainerman arXiv:0904.0982 [gr-qc]

Theorem (Alexakis, Ionescu, Klainerman 2009)

Regular non-degenerate *vacuum black holes* *near non-extreme Kerr* *are Kerr*

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Computer assisted proof using the *Jezierski-Kamiński* potentials: reduce the problem to a (numerical) proof that *explicit linear Fuchsian ODEs on* $(-1, 1)$ *for eight complex-valued functions* W_k, $k = 1, \ldots, 8$ have no bounded solutions

$k \neq 0$, second order equation for W_k

- $W_k(x)$ satisfies

$$8\,(x-1)^2(x+1)^2\left(x^2+1\right)\Big[1/2\,k^2x^8+k\,(i+2\,k)\,x^6+\left(i\,k+3\,k^2-8\right)x^4+\left(-ik+2\,k^2+16\right)x^2$$

$$-ik+1/2\,k^2-8\Big]\,W_k''(x)$$

$$+16\,(x+1)\,(x-1)\Big[k\,(i+k)\,x^8+\left(4\,k^2-16\right)x^6+\left(-2\,ik+6\,k^2+64\right)x^4+\left(4\,k^2-80\right)x^2+ik+k^2+32\Big]\,xW_k'(x)$$

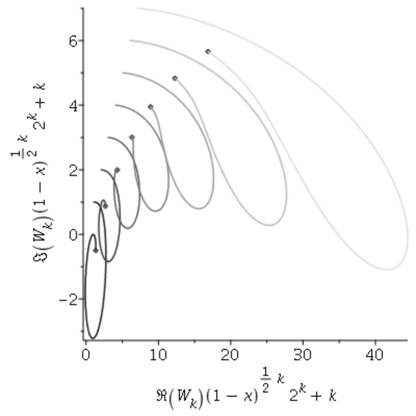
$$-2\left[1/2\,k^4x^{14}+k^3\,(i+7/2\,k)\,x^{12}+4\,k\left(ik^2+\frac{21\,k^3}{8}-6\,i-11\,k\right)x^{10}+\right.$$

$$\left(-68\,k^2+64+5\,ik^3+\frac{35\,k^4}{2}-40\,ik\right)x^8+\left(40\,k^2-128+\frac{35\,k^4}{2}+240\,ik\right)x^6$$

$$+\left(88\,k^2+256-5\,ik^3+21/2\,k^4+16\,ik\right)x^4+\left(4\,k^2-384-4\,ik^3+7/2\,k^4-216\,ik\right)x^2$$

$$\left.-\left(k^2-24\right)\left(ik-1/2\,k^2+8\right)\right]\,W_k(x)=0$$

Numerical solutions: regular solutions would produce closed curves



Stability, $\Lambda = 0$: (on the 25th anniversary of the Christodoulou-Klainerman theorem, almost there...)

Dafermos, Imperial College Colloquium, January 2018

Theorem (Dafermos, Holzegel, Rodnianski, Taylor; in preparation)

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Theorem (Klainerman, Szeftel; in preparation)

Schwarzschild black holes are stable under axi-symmetric polarised non-linear perturbations

$\Lambda < 0$: things are different

- Existence of infinite dimensional families of globally well behaved stationary solutions
- Instabilities

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$\Lambda < 0$: the global structure of anti de Sitter space-time

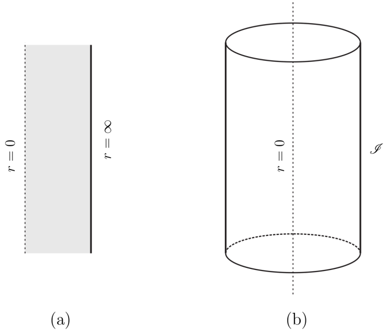

Figure 5.2.8: The conformal structure of anti-de Siter spacetime. The two-dimensional projection is the shaded strip $0 \leq r < \infty$ of figure (a). Since $\{r = 0\}$ is a center of rotation, a more faithful representation is provided by the solid cylinder of figure (b).

$\Lambda < 0$: the global structure of Schwarzschild-anti de Sitter metrics

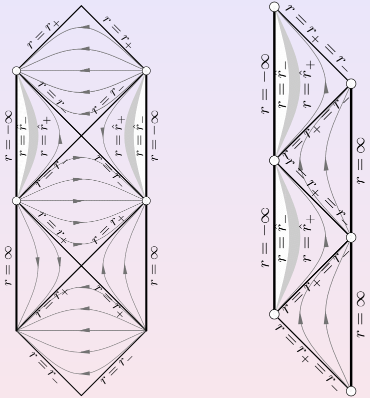

Figure: The projection diagrams for the Kerr-Newman - anti de Sitter metrics with two distinct zeros of Δ_r (left diagram) and one double zero (right diagram).

$\Lambda < 0$: Instability

Formation of singularities in finite time with arbitrarily small perturbations

Bizoń and Rostworowski (2011):
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Theorem (Moschidis, in preparation)

Spherically symmetric Einstein-Vlasov equations *are unstable near the anti-de Sitter space-time*

$\Lambda < 0$: Static globally regular vacuum black holes

3+1 dim.: 2002; n+1 dim.: 2005

Theorem (Anderson, PTC, Delay 2002, 2005)

There exists an infinite dimensional family of globally regular static vacuum black holes with $\Lambda < 0$ near SAdS

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true for large classes of Riemannian *Birmingham metrics*

$$g = +(r^2 + \kappa - \frac{2m}{r^{n-2}})dt^2 + \frac{dr^2}{r^2 + \kappa - \frac{2m}{r^{n-2}}} + r^2 h$$

where (N^{n-1}, h) satisfies $R_{AB}(h) = k(n-2)h_{AB}$

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• Construct complex-valued tensor and matter fields, parameterised by a *complex-parameter a*, solutions of elliptic equations obtained by a *Wick rotation*

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3. real-valued and Lorentzian for real a after undoing the Wick rotation.

$\Lambda < 0$: Stationary globally regular black holes with or without matter fields, example of arguments

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$\Lambda < 0$: Stationary globally regular vacuum black holes , example of arguments

For the purpose of illustration, let us assume vacuum

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• Let $(x^i) = (x, x^A)$, $A = 2, \ldots, n$, and consider a Lorentzian non-degenerate static vacuum metric on $\mathbb{R} \times \Sigma$, in local coordinates (t, x^i),

$$\mathring{g} = x^{-2}(dx^2 + \mathring{\bar{g}}_{00}(x^i)dt^2 + \mathring{\bar{g}}_{AB}(x^i)dx^A dx^B)$$

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• We wish to construct one-parameter families of vacuum metrics of the form

$$g = x^{-2}(dx^2 + \bar{g}_{00}(a, x^i)dt^2 + 2\bar{g}_{0A}(a, x^i)dtdx^A + \bar{g}_{AB}(a, x, x^C)dx^A dx^B)$$

with asymptotics

$$\bar{g}_{00}(a, x^C) \to_{x\to 0} \mathring{\bar{g}}_{00}(x^C)\,, \quad \bar{g}_{AB}(a, x^C) \to_{x\to 0} \mathring{\bar{g}}_{AB}(x^C)\,,$$
$$\bar{g}_{0A}(a, x^C) \to_{x\to 0} a\psi_A(x^C)\,,$$

with *given* $\psi_A(x^C)$, a, with a real and small .

$\Lambda < 0$: Stationary globally regular vacuum black holes , example of arguments

- Let P denote the linearisation of

$$R_{\mu\nu} - \lambda g_{\mu\nu} \equiv 0 \quad (1)$$

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on $S^1 \times \Sigma$

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on $S^1 \times \Sigma$

- For all small complex a the implicit function theorem gives complex-valued solutions of (1)

$$g = x^{-2}(dx^2 - \bar{g}_{00}(a, x^i)dt^2 + 2i\bar{g}_{0A}(a, x^i)dtdx^A + \bar{g}_{AB}(a, x^i)dx^A dx^B)$$

with asymptotics

$$\bar{g}_{00}(a, x^i) \to_{x\to 0} \mathring{\bar{g}}_{00}(x^C), \quad \bar{g}_{AB}(a, x^i) \to_{x\to 0} \mathring{\bar{g}}_{AB}(x^C),$$
$$\bar{g}_{0A}(a, x^i) \to_{x\to 0} a\psi_A(x^C).$$

universität wien

$\Lambda < 0$: Stationary globally regular vacuum black holes , example of arguments

- Write $a = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, and let

$$\partial_{\overline{a}} := \partial_\alpha + i\partial_\beta$$

be the usual complex derivative operator with respect to the complex conjugate $\overline{a}$ of a.

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- Since g solves the Einstein equation, $\partial_{\bar{a}} g$ satisfies

$$0 = \partial_{\bar{a}}\big(R_{\mu\nu} - \lambda g_{\mu\nu}\big) = P\partial_{\bar{a}} g_{\mu\nu} \; ,$$

with vanishing asymptotic data, hence

$$\partial_{\bar{a}} g \equiv 0$$

since P has no kernel.

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- Thus g is holomorphic in a.

$\Lambda < 0$: Stationary globally regular vacuum black holes , example of arguments

• Since g solves the Einstein equation, $\partial_t g$ satisfies

$$0 = \partial_t \big(R_{\mu\nu} - \lambda g_{\mu\nu} \big) = P \partial_t g_{\mu\nu} \;,$$

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• For $a = i\beta$, $\beta \in \mathbb{R}$, the boundary conditions are real, and there exists a (real-valued) Riemannian solution, by uniqueness $g(i\beta)$ is a Riemannian metric.

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$\Lambda < 0$: Stationary globally regular vacuum black holes , example of arguments

• Let ϕ be the time-reversal, $t \mapsto -t$. Then $\phi^* g(-a)$ and $g(a)$ satisfy the same equations and same bouundary conditions. By uniqueness

$$\phi^* g(-a) = g(a)\,.$$

Equivalently

$$g_{ti}(-a) = -g_{ti}(a)\,, \quad g_{tt}(-a) = g_{tt}(a)\,, \quad g_{ij}(-a) = g_{ij}(a)\,. \qquad (2)$$

• For $a \in \mathbb{R}$, (2) and the fact that $g(i\beta)$ is a Riemannian metric show that

$$g_{tk}(a) \in i\mathbb{R}\,, \quad g_{tt}(a) \in \mathbb{R}^-\,, \quad g_{k\ell}(a) \in \mathbb{R}\,. \qquad (3)$$

e.g.:

$$g_{tk}(a) = \sum_{n\in\mathbb{N}} (g_{tk})_n a^n = \sum_{n=2k+1\,,\,k\in\mathbb{N}} \underbrace{(g_{tk})_n}_{\in i\mathbb{C}} a^{2k+1}$$

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$\Lambda < 0$: Stationary globally regular vacuum black holes, example of arguments

- This proves that the replacement $t \mapsto it$ provides a one-parameter family of Lorentzian (real-valued) metric solving the vacuum Einstein equations with a negative cosmological constant for all small $a \in \mathbb{R}$.

$\Lambda > 0$: the global structure of de Sitter space-times

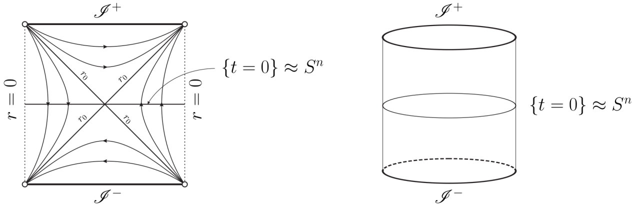

Figure 5.3.4: The generalized Kottler (de Sitter) metrics with positive cosmological constant and vanishing mass parameter m. Left figure: a conformal diagram; the lines $\{r = 0\}$ are centers of rotation. The right figure makes it clearer that the Cauchy surface $\{t = 0\}$, as well as $\mathscr{I}^+$ and $\mathscr{I}^-$, have spherical topology.

$\Lambda > 0$: the global structure of Schwarzschild-de Sitter metrics

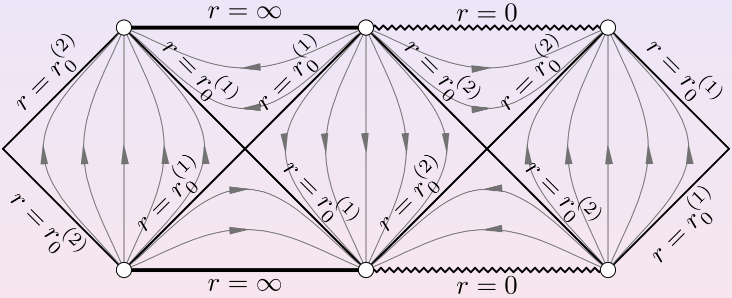

Figure: conformal diagram for Schwarzschild-de Sitter metrics

$\Lambda > 0$: the global structure of Kerr-de Sitter space-times

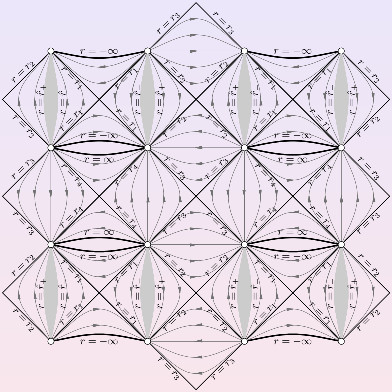

Figure: A projection diagram for the Kerr-de Sitter spacettimes with four distinct zeros of $\Delta_r = (r^2 + a^2)\left(1 - \frac{\Lambda}{3}r^2\right) - 2m(1 + \frac{\Lambda}{3}a^2)r$.

$\Lambda > 0$: Stability (32 years of Friedrich's theorem ...)

Theorem (Friedrich 1986)

de Sitter is stable under non-linear perturbations

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Theorem (Friedrich 1986)

de Sitter is stable under non-linear perturbations

Theorem (Hintz, Vasy 2017)

Kerr-de Sitter black holes are stable under non-linear perturbations in the region between horizons

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compare Schlue (work in progress) and Ringström, Oxford Univ. Press 2013

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compare Schlue (work in progress) and Ringström, Oxford Univ. Press 2013

Corollary (Hintz 2018)

Stationary vacuum black holes near Schwarzschild-de Sitter are Kerr-de Sitter between the Killing horizons.

$\Lambda > 0$: Uniqueness, static case

$$g = -V^2 dt^2 + g_{ij} dx^i dx^j \,, \quad \partial_t V = 0 = \partial_t g_{ij} \,.$$

Theorem (Lafontaine, Rozoy 1999)

Schwarzschild-de Sitter black holes are unique *in the class of* static *black holes with* nice *level sets of the "lapse function" V*

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Theorem (Borghini, Mazzieri 2017)

Schwarzschild-de Sitter's are the only static black holes

near Schwarzschild-de Sitter

and satisfying a "virtual mass" condition

(special case of a more general theorem)

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