Four-Graviton Scattering and String Path Integral in the Proper-time Gauge

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USES of String Field Theory and Polyakov string path integrals

I will introduce the covariant string field theory and the Polyakov string path integrals in the proper-time gauge. In the proper-time gauge, the string path integral can be written as integrals over the proper-times in a way similar to the Schwinger's proper time representation of Feynman integrals of quantum field theory. For this reason, it becomes feasible in the proper-time gauge to identify the field theoretical expressions of the string path integrals which depict multiple string scatterings. The four-graviton scattering will be evaluated and compared with the conventional one obtained by using the vertex operators. The string field theory may be useful to explore the quantum gravity in the region of off-shell.

Quantum Gravity and String Field Theory

Classical General Relativity Derived from Quantum Gravity

Boulware and Deser, Ann. Phys. 89 (1975):
"A quantum particle description of local (noncosmological) gravitational phenomena necessarily leads to a classical limit which is just a metric theory of gravity. As long as only helicity ± 2 gravitons are included, the theory is precisely Einsteins general relativity."

Closed String Field Theory

Closed string theory contains massless spin 2 particles in its spectrum. The low energy limit of the covariant interacting closed string field theory must be the Einstein's general relativity. The closed string field theory may provide a consistent framework to describe a finite quantum theory of the spin 2 particles, the gravitons. We need to examine the graviton scattering amplitudes of the covariant string field theory and compare them with those of the perturbation theory of the gravity in the low energy region.

Gravity as a low energy limit of a closed string field theory

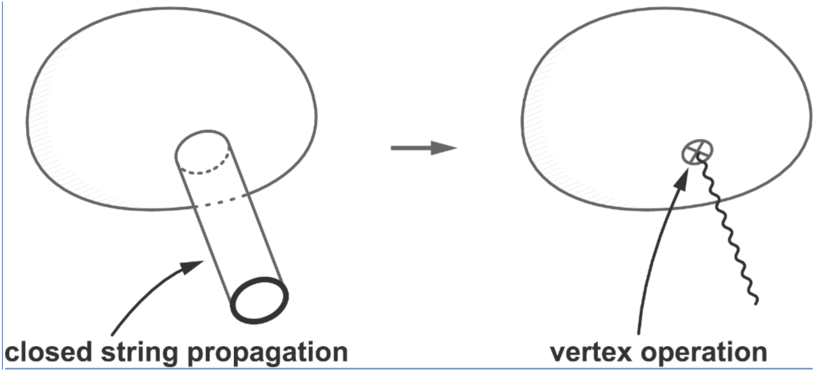
Quantum gravity and closed string field theory

1. Closed string field theory may be finite: It may resolve the problem associated with non-renormalizability of the perturbative quantum gravity
2. Closed string field theory is unitary: It may provide a resolution to the problems related to the information loss and the Bekenstein-Hawking entropy of black holes
3. Closed string field theory may have well-defined UV and IR behaviors.

Vertex Operators

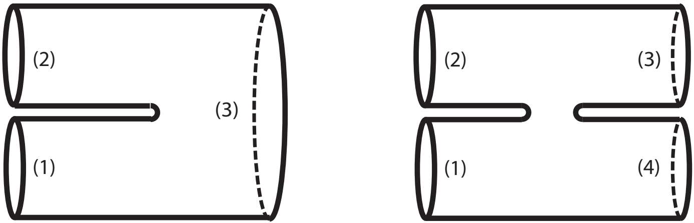
String Scattering Amplitudes with Vertex Operators

String scattering amplitudes with vertex operators may not be valid in the off-shell and may not capture the correct UV behavior of string theory.



II. Closed string field theory in the proper-time gauge

We will construct a covariant string field theory and calculate three-string scattering amplitude and the four-string scattering amplitude in the low energy limit by extending the previous works on the open string field theory.



Fock Space Representation of the Closed String Field Theory in the Proper-Time Gauge

Closed String Field Theory in the Proper-Time Gauge

$$S = \langle \Phi | \mathcal{K} \Phi \rangle + \frac{g}{3} \Big(\langle \Phi | \Phi \circ \Phi \rangle + \langle \Phi \circ \Phi | \Phi \rangle \Big).$$

The closed string field theory in the proper-time gauge generates the string scattering diagrams, which can be represented by the Polyakov string path integrals:

$$S_P = -\frac{1}{4\pi\alpha'} \int_M d\tau d\sigma \sqrt{-h} h^{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu}, \qquad \mu, \nu = 0, \dots, d-1.$$

String Scattering Amplitudes

$$\mathcal{A}_M = \int D[X] D[h] \exp \left[-i \int_M d\tau d\sigma \mathcal{L} \right].$$

String Scattering Amplitudes of Closed String Field Theory and Polyakov String Path Integral

Strategy of Calculation of String Scattering Amplitudes

1. Evaluate the scattering amplitudes by using the Polyakov string path integral
2. Re-express the Polakov string path integrals in terms of the oscillator operators
3. Identify the Fock space (operator) representations of the string field theory vertices
4. Choose appropriate external string states, corresponding to the various particle states and evaluate the scattering amplitudes.
5. Compare the resultant scattering amplitudes with those of YM theory for open string and GR for closed string in the zero-slope limit

Closed String Theory: Review

Free String Theory

$$S_= \frac{1}{4\pi\alpha'}\int d\tau d\sigma\, \partial X \cdot \partial X.$$

Decomposition of X in terms of left-movers and right-movers

$$X(\tau,\sigma) = X_L(\tau+\sigma) + X_R(\tau-\sigma).$$

Mode expansions

$$\begin{aligned} X_L(\tau,\sigma) &= x_L + \sqrt{\frac{\alpha'}{2}}\, p_L(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in(\tau+\sigma)}, \\ X_R(\tau,\sigma) &= x_R + \sqrt{\frac{\alpha'}{2}}\, p_R(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n e^{-in(\tau-\sigma)}, \end{aligned}$$

where $x = x_L + x_R$.

Closed String Theory: Review

Canonical commutation relations

$$
\begin{aligned}
[x_L, p_L] &= [x_R, p_R] = i\sqrt{\frac{\alpha'}{2}}, \\
[\alpha_m, \alpha_n] &= [\tilde{\alpha}_m, \tilde{\alpha}_n] = m\delta(m+n).
\end{aligned}
$$

Momentum eigentate with eigenvalue $P_n, n \neq 0$:

$$
|P_n\rangle = \sqrt{\frac{1}{\pi n}} \exp\left\{ \left(\frac{P_n \alpha_{-n}}{n} + \frac{P_{-n}\tilde{\alpha}_{-n}}{n} - \frac{\alpha_{-n}\tilde{\alpha}_{-n}}{n} - \frac{P_n \cdot P_{-n}}{4n} \right) \right\} |0\rangle.
$$

Mapping from cylindrical surface onto the complex plane

$$
z = e^{\rho} = e^{\xi + i\eta}, \qquad -\pi \leq \eta \leq \pi.
$$

Green's function on complex plane ($\xi > \xi'$), $\Delta = |\xi - \xi'|$,

$$
\begin{aligned}
G_C(z, z') &= \ln |z - z'| \\
&= \max(\xi, \xi') - \frac{1}{2} \sum_{n=1} \frac{e^{-n\Delta}}{n} \left(e^{in(\eta' - \eta)} + e^{-in(\eta' - \eta)} \right).
\end{aligned}
$$

Closed String Interaction in the Proper Time Gauge

CS mapping from the world sheet of three closed string scattering onto the complex plane. For the three-string vertex in the proper-time gauge,

$$\rho = \ln(z-1) + \ln z.$$

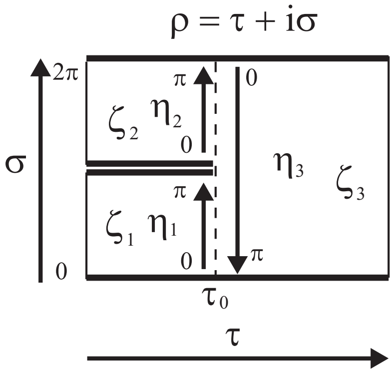
The local coordinates $\zeta_r = \xi_r + i\eta_r$, $r = 1, 2, 3$ defined on invidual string world sheet patches are related to z as follows:

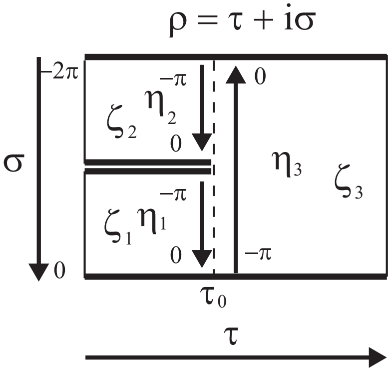
$$\begin{aligned} e^{-\zeta_1} &= e^{\tau_0} \frac{1}{z(z-1)}, \\ e^{-\zeta_2} &= -e^{\tau_0} \frac{1}{z(z-1)}, \\ e^{-\zeta_3} &= -e^{-\frac{\tau_0}{2}} \sqrt{z(z-1)}. \end{aligned}$$

For convenience we choose, by using $SL(2, C)$ invariance

$$Z_1 = 0, \quad Z_2 = 1, \quad Z_3 = \infty.$$

Local Coordinates





Neumann Functions of the Closed String Vertices

Fourier components of the Green's function on complex plane

$$\begin{aligned} G_C(\rho_r, \rho'_s) &= \ln |z_r - {z'}_s| \\ &= -\delta_{rs} \left\{ \sum_{n=1} \frac{e^{-n\Delta}}{2n} \left(e^{in(\eta'_s - \eta_r)} + e^{-in(\eta'_s - \eta_r)} \right) - \max(\xi, \xi') \right\} \\ &\quad + \sum_{n,m} \bar{C}^{rs}_{nm} e^{|n|\xi_r + |m|\xi'_s} e^{in\eta_r} e^{im\eta'_s}. \end{aligned}$$

Integral Formulas for $\bar{C}^{rs}_{nm}$

$$
\begin{aligned}
\bar{C}^{rs}_{00} &= \ln|Z_r - Z_s|, \quad r \neq s, \\
\bar{C}^{rr}_{00} &= -\sum_{i \neq r} \frac{\alpha_i}{\alpha_r} \ln|Z_r - Z_i| + \frac{1}{\alpha_r}\tau^{(r)}_0 \\
\bar{C}^{rs}_{n0} &= \bar{C}^{sr}_{0n} = \frac{1}{2n}\oint_{Z_r} \frac{dz}{2\pi i} \frac{1}{z - Z_s} e^{-n\zeta_r(z)}, \quad n \geq 1, \\
\bar{C}^{rs}_{nm} &= \frac{1}{2nm}\oint_{Z_r} \frac{dz}{2\pi i} \oint_{Z_s} \frac{dz'}{2\pi i} \frac{1}{(z - z')^2} e^{-n\zeta_r(z) - m\zeta'_s(z')}, \quad n, m \geq 1.
\end{aligned}
$$

Reality conditions of the Green's function

$$
\bar{C}^{rs}_{nm} = \bar{C}^{*rs}_{-n-m}, \quad \bar{C}^{rs}_{-nm} = 0, \quad n, m \geq 1.
$$

Scattering amplitude

$$
\begin{aligned}
\mathcal{W} &= \int DX \exp\left(i \sum_{r=1}^{M} \int P_r(\sigma)\cdot X(\tau_r,\sigma) d\sigma - \int d\tau d\sigma \mathcal{L} \right) \\
&= [\det \Delta]^{-d/2} \exp\left\{ \frac{1}{4} \left\{ \sum_r \xi_r P_0^2 - \sum_r \sum_{n=1} \frac{1}{n} P_n^{(r)} \cdot P_{-n}^{(r)} \right.\right. \\
&\quad \left.\left. + \sum_{n,m} \bar{C}^{rs}_{nm} e^{|n|\xi_r + |m|\xi'_s} P_{-n}^{(r)} \cdot P_{-m}^{(s)} \right\} \right. \\
&= \langle \mathbf{P} | \exp\left(\sum_r \xi_r L_0^{(r)} \right) | V[N] \rangle .
\end{aligned}
$$

Neumann Functions of Three-Closed-String Vertex

Neumann functions of three-closed string vertex and those of three-open string vertex:

$$
\begin{aligned}
\bar{C}^{rs}_{00} &= \bar{N}^{rs}_{00} = \ln|Z_r - Z_s|, \quad r \neq s, \\
\bar{C}^{rr}_{00} &= \bar{N}^{rr}_{00} = -\sum_{i\neq r} \frac{\alpha_i}{\alpha_r} \ln|Z_r - Z_i| + \frac{1}{\alpha_r}\tau_0, \\
\bar{C}^{rs}_{n0} &= \bar{C}^{rs}_{-n0} = \frac{1}{2}\bar{N}^{rs}_{n0} = \frac{1}{2n}\oint_{Z_r} \frac{dz}{2\pi i}\frac{1}{z - Z_s} e^{-n\zeta_r(z)}, \quad n \geq 1, \\
\bar{C}^{rs}_{nm} &= \bar{C}^{rs}_{-n-m} = \frac{1}{2}\bar{N}^{rs}_{mn} \\
&= \frac{1}{2nm}\oint_{Z_r}\frac{dz}{2\pi i}\oint_{Z_s}\frac{dz'}{2\pi i}\frac{1}{(z-z')^2} e^{-n\zeta_r(z) - m\zeta'_s(z')}, \quad n, m \geq 1, \\
\bar{C}^{rs}_{n-m} &= \bar{C}^{rs}_{-nm} = 0.
\end{aligned}
$$

Factorization of Three-Closed-String Scattering Amplitude

$$
\begin{aligned}
\mathcal{A}[1,2,3] \quad = \quad & g\langle\{\mathbf{k}^{(r)}\}| \\
& \exp\left\{\sum_{r,s}\Big(\sum_{n,m\geq 1}\frac{1}{2}\bar{N}^{rs}_{nm}\frac{\alpha^{(r)\dagger}_{n}}{2}\cdot\frac{\alpha^{(r)\dagger}_{m}}{2}+\sum_{n\geq 1}\bar{N}^{rs}_{n0}\frac{\alpha^{(r)\dagger}_{n}}{2}\cdot\frac{p^{(s)}}{2}\Big)\right\} \\
& \exp\left\{\tau_0\sum_{r}\frac{1}{\alpha_r}\left(\frac{1}{2}\left(\frac{p^{(r)}}{2}\right)^2-1\right)\right\} \\
& \exp\left\{\sum_{r,s}\Big(\sum_{n,m\geq 1}\frac{1}{2}\bar{N}^{rs}_{nm}\frac{\tilde{\alpha}^{(r)\dagger}_{n}}{2}\cdot\frac{\tilde{\alpha}^{(r)\dagger}_{m}}{2}+\sum_{n\geq 1}\bar{N}^{rs}_{n0}\frac{\tilde{\alpha}^{(r)\dagger}_{n}}{2}\cdot\frac{p^{(s)}}{2}\Big)\right\} \\
& \exp\left\{\tau_0\sum_{r}\frac{1}{\alpha_r}\left(\frac{1}{2}\left(\frac{p^{(r)}}{2}\right)^2-1\right)\right\}|0\rangle.
\end{aligned}
$$

Three-Closed-String Scattering Amplitude

- Three-Closed-String Scattering Amplitude $=$
  three-tachyon scattering amplitude
  $+$ two-tachyon and one graviton scattering amplitude
  $+$ one tachyon and two-graviton scattering amplitude
  $+$ three-graviton scattering amplitude
  $+ \dots$
- Valid in the region of off-shell.
- Non-Locality (off-shell).

Factorization of Three-Closed-String Scattering Amplitude

Factorization of three-closed-string scattering amplitude

$$\mathcal{A}_{\text{closed}}[1,2,3] = \mathcal{A}_{\text{open}}[1,2,3]\,\mathcal{A}_{\text{open}}[1,2,3].$$

Scattering amplitude of three closed strings can be completely factorized into those of three open strings except for the zero modes.

Question: Can we factorize general closed string scattering amplitudes into those of open string theory?

Realization of the Kawai-Lewellen-Tye (KLT) relations in the framework of the second quantized string theory .

Three-Graviton Scattering Amplitude

Decomposition of the spin-2 field into graviton, anti-symmetric tensor, and scalar field

$$h_{\mu\nu} = \left\{\frac{1}{2}\left(h_{\mu\nu} + h_{\nu\mu}\right) - \eta_{\mu\nu}\frac{1}{d}h^{\sigma}{}_{\sigma}\right\} + \left\{\frac{1}{2}\left(h_{\mu\nu} - h_{\nu\mu}\right)\right\} + \eta_{\mu\nu}\left\{\frac{1}{d}h^{\sigma}{}_{\sigma}\right\}.$$

We choose the covariant gauge condition

$$\partial^{\mu} h_{\mu\nu} = 0,$$

which becomes de Donder gauge condition for the graviton

$$\partial^{\mu} h_{\mu\nu} - \frac{1}{d-2}\partial_{\nu} h^{\sigma}{}_{\sigma} = 0.$$

For three-graviton scattering, we choose the external string state as

$$|\Psi_{3G}\rangle = \prod_{r=1}^{3}\left\{h_{\mu\nu}(p^r)\alpha^{(r)\mu}_{-1}\tilde{\alpha}^{(r)\nu}_{-1}\right\}|0\rangle.$$

Three-Graviton Scattering Amplitude

Three-graviton scattering amplitude

$$
\begin{aligned}
\mathcal{A}_{[3-graviton]} &= \int \prod_{r=1}^{3} dp^{(r)} \delta \left(\sum_{r=1}^{3} p^{(r)} \right) \frac{2g}{3} \langle \Psi_{3G} | E^{\text{Closed}}_{[3]}[1,2,3] | 0 \rangle \\
&= \left(\frac{2g}{3} \right) e^{-2\tau_0 \sum_{r=1}^{3} \frac{1}{\alpha_r}} \int \prod_{i=1}^{3} dp^{(i)} \delta \left(\sum_{i=1}^{3} p^{(i)} \right) \\
& \langle 0 | \left\{ \prod_{i=1}^{3} h_{\mu\nu}(p^{(i)}) a_1^{(i)\mu} \cdot \tilde{a}_1^{(i)\nu} \right\} \frac{1}{2^5} \left(\sum_{r,s=1}^{3} \bar{N}^{rs}_{11} a_1^{(r)\dagger} \cdot a_1^{(s)\dagger} \right) \\
& \left(\sum_{t=1}^{3} \bar{N}^{t}_{1} a_1^{(t)\dagger} \cdot \mathbf{p} \right) \frac{1}{2^5} \left(\sum_{l,m=1}^{3} \bar{N}^{lm}_{11} \tilde{a}_1^{(l)\dagger} \cdot \tilde{a}_1^{(m)\dagger} \right) \\
& \left(\sum_{n=1}^{3} \bar{N}^{n}_{1} \tilde{a}_1^{(n)\dagger} \cdot \mathbf{p} \right) | 0 \rangle .
\end{aligned}
$$

Three-Graviton Scattering Amplitude

$\mathcal{A}_{[3-graviton]}$ is precisely the three-graviton interaction term which may be obtained from the Einstein's gravity action.

$$\begin{aligned}\mathcal{A}_{[3-graviton]} &= \kappa \int \prod_{i=1}^{3} dp^{(i)} \delta\left(\sum_{i=1}^{3} p^{(i)}\right) \\ & \quad h_{\mu_1\nu_1}(p^{(1)}) h_{\mu_2\nu_2}(p^{(2)}) h_{\mu_3\nu_3}(p^{(3)}) \\ & \quad \left\{\eta^{\mu_1\mu_2} p^{(1)\mu_3} + \eta^{\mu_2\mu_3} p^{(2)\mu_1} + \eta^{\mu_3\mu_1} p^{(3)\mu_2}\right\} \\ & \quad \left\{\eta^{\nu_1\nu_2} p^{(1)\nu_3} + \eta^{\nu_2\nu_3} p^{(2)\nu_1} + \eta^{\nu_3\nu_1} p^{(3)\nu_2}\right\}\end{aligned}$$

where $\kappa = \frac{g}{2^7 \cdot 3} = \sqrt{32\pi G_{10}}$.

Scattering Amplitude of Four Strings

Using the Cremmer-Gervais identity, we may write the scattering amplitude of four closed strings as follows

$$\mathcal{A}[1,2,3,4] = g^2 \int \prod_r dZ_r^2 \frac{|Z_a - Z_b|^2 |Z_b - Z_c|^2 |Z_c - Z_a|^2}{d^2 Z_a d^2 Z_b d^2 Z_c}$$

$$\prod_{r<s} |Z_r - Z_s|^{2\,\frac{p^{(r)}}{2}\cdot\frac{p^{(s)}}{2}} \prod_{r<s} |Z_r - Z_s|^{2\left\{\frac{\alpha_s}{\alpha_r}\left(1-\frac{1}{2}\left(\frac{p^{(r)}}{2}\right)^2\right)+\frac{\alpha_r}{\alpha_s}\left(1-\frac{1}{2}\left(\frac{p^{(s)}}{2}\right)^2\right)\right\}}$$

$$\langle\{\mathbf{k}^{(r)}\}|\exp\left\{\frac{1}{4}\sum_{r,s}\sum_{n,m\geq 1}\left(\bar{C}^{rs}_{nm}\,\tilde{\alpha}^{(r)\dagger}_n\cdot\tilde{\alpha}^{(s)\dagger}_m + \bar{C}^{rs*}_{nm}\,\alpha^{(r)\dagger}_n\cdot\alpha^{(r)\dagger}_m\right)\right.$$

$$+\sum_{r,s}\left(\sum_{n\geq 1}\left(\bar{C}^{rs}_{n0}\tilde{\alpha}^{(r)\dagger}_n\cdot\frac{p^{(s)}}{2} + \bar{C}^{rs*}_{n0}\,\alpha^{(r)\dagger}_n\cdot\frac{p^{(s)}}{2}\right)\right)$$

$$\left.+2\tau^{(r)}\sum_r\frac{1}{\alpha_r}\left(\frac{1}{2}\left(\frac{p^{(r)}}{2}\right)^2 - 1\right)\right\}|0\rangle.$$

Shapiro-Virasoro Amplitude

Scattering of four closed string tachyons

$$\begin{aligned}
\mathcal{A}_{\text{Tachyon}}[1,2,3,4] &= g^2 \int d^2Z\, |Z|^{2\,\frac{p^{(1)}}{2}\cdot\frac{p^{(2)}}{2}} |1-Z|^{2\,\frac{p^{(2)}}{2}\cdot\frac{p^{(3)}}{2}} \\
&= g^2 \int d^2Z\, |Z|^{2\left(-\frac{s}{8}-2\right)} |1-Z|^{2\left(-\frac{u}{8}-2\right)} \\
&= 2\pi g^2 \frac{\Gamma\left(-1-\frac{s}{8}\right)\Gamma\left(-1-\frac{t}{8}\right)\Gamma\left(-1-\frac{u}{8}\right)}{\Gamma\left(2+\frac{s}{8}\right)\Gamma\left(2+\frac{t}{8}\right)\Gamma\left(2+\frac{u}{8}\right)}.
\end{aligned}$$

Mandelstam variables:

$$s = -(p_1+p_2)^2, \quad t = -(p_1+p_3)^2, \quad u = -(p_1+p_4)^2.$$

The Koba-Nielsen variables:

$$Z_1 = 0, \quad Z_2 = Z, \quad Z_3 = 1, \quad Z_4 = \infty.$$

Local coordinates for four-closed-string scattering

Schwarz-Christoffel transformation

$$\rho = \sum_{r=1}^{4} \alpha_r \ln(z - Z_r) = \ln z + \ln(z - Z) - \ln(z - 1) + i\pi.$$

Equivalently, $e^\rho = -\frac{z(z-Z)}{z-1}$. Interaction points on the complex plane are determined by

$$\frac{\partial \rho}{\partial z} = \sum_{r=1}^{4} \frac{\alpha_r}{z - Z_r} = 0$$

which has two solutions:

$$z_\pm = 1 \pm \sqrt{1 - Z}.$$

These two solutions define two interaction points

$$\rho(z_\pm) = \ln \frac{z_\pm(z_\pm - Z)}{z_\pm - 1} + i\pi$$

Local coordinate patches

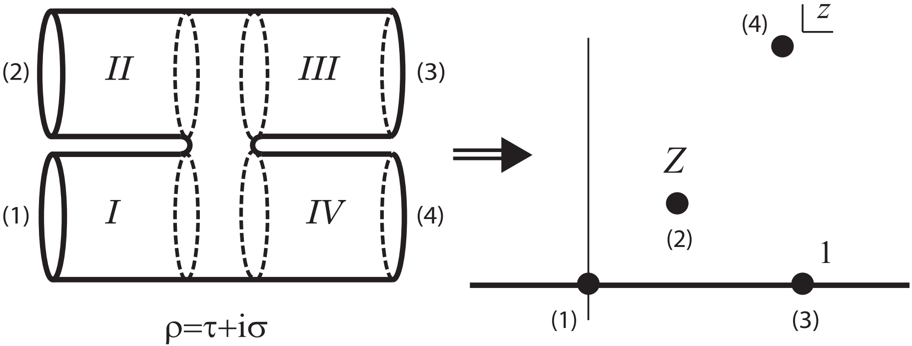
Two interaction points on the world sheet:

$$
\begin{aligned}
\tau_1 &= 2\,\mathrm{Re}\,\ln\left(1-\sqrt{1-Z}\right), \quad \tau_2 = 2\,\mathrm{Re}\,\ln\left(1+\sqrt{1-Z}\right), \\
\sigma_1 &= 2\,\mathrm{Im}\,\ln\left(1-\sqrt{1-Z}\right)+\pi, \quad \sigma_2 = 2\,\mathrm{Im}\,\ln\left(1+\sqrt{1-Z}\right)+\pi.
\end{aligned}
$$

Global coordinates on the local patches:

$$
\rho = \begin{cases}
\zeta_1+\tau_1+i\sigma_1-i\pi, & \text{on} \quad I \\
\zeta_2+\tau_1+i\sigma_1, & \text{on} \quad II \\
-\zeta_3+\tau_2+i\sigma_2+i\pi, & \text{on} \quad III \\
-\zeta_4+\tau_2+i\sigma_2, & \text{on} \quad IV.
\end{cases}
$$

Local coordinate patches



Conformal mapping from the string world-sheet to the complex plane

The conformal mapping from the local string patches to the complex plane may be written as follows :

$$e^{-\zeta_1} = e^{\tau_1 + i\sigma_1} \frac{(z-1)}{z(z-Z)}, \qquad \text{on the 1st string patch,}$$

$$e^{-\zeta_2} = -e^{\tau_1 + i\sigma_1} \frac{(z-1)}{z(z-Z)}, \qquad \text{on the 2nd string patch,}$$

$$e^{-\zeta_3} = e^{-\tau_2 - i\sigma_2} \frac{z(z-Z)}{z-1}, \qquad \text{on the 3rd string patch,}$$

$$e^{-\zeta_4} = -e^{-\tau_2 - i\sigma_2} \frac{z(z-Z)}{z-1}, \qquad \text{on the 4th string patch.}$$

Neuumann functions for four-closed-string scattering

$$\bar{C}^{rs}_{n0} \;=\; \bar{C}^{sr}_{0n} = \frac{1}{2n}\oint_{Z_r}\frac{dz}{2\pi i}\frac{1}{z-Z_s}e^{-n\zeta_r(z)}, \quad n \ge 1,$$

$$\begin{aligned}
\bar{C}^{11}_{10} &= \frac{e^{\tau_1+i\sigma_1}}{2}\frac{1-Z}{Z^2}, \quad \bar{C}^{12}_{10} = -\frac{e^{\tau_1+i\sigma_1}}{2}\frac{1}{Z^2}, \quad \bar{C}^{13}_{10} = -\frac{e^{\tau_1+i\sigma_1}}{2}\frac{1}{Z}, \\
\bar{C}^{14}_{10} &= 0, \quad \bar{C}^{21}_{10} = \frac{e^{\tau_1+i\sigma_1}}{2}\frac{1-Z}{Z^2}, \quad \bar{C}^{22}_{10} = -\frac{e^{\tau_1+i\sigma_1}}{2}\frac{1}{Z^2}, \\
\bar{C}^{23}_{10} &= -\frac{e^{\tau_1+i\sigma_1}}{2}\frac{1}{Z}, \quad \bar{C}^{24}_{10} = 0, \quad \bar{C}^{31}_{10} = \frac{e^{-\tau_2-i\sigma_2}}{2}(1-Z), \\
\bar{C}^{32}_{10} &= \frac{e^{-\tau_2-i\sigma_2}}{2}, \quad \bar{C}^{33}_{10} = \frac{e^{-\tau_2-i\sigma_2}}{2}(2-Z), \quad \bar{C}^{34}_{10} = \bar{C}^{44}_{10} = 0, \\
\bar{C}^{41}_{10} &= \frac{e^{-\tau_2-i\sigma_2}}{2}(1-Z), \quad \bar{C}^{42}_{10} = \frac{e^{-\tau_2-i\sigma_2}}{2}, \quad \bar{C}^{43}_{10} = \frac{e^{-\tau_2-i\sigma_2}}{2}(2-Z).
\end{aligned}$$

Neuumann functions for four-closed-string scattering

$$\bar{C}^{rs}_{nm} = \frac{1}{2nm}\oint_{Z_r}\frac{dz}{2\pi i}\oint_{Z_s}\frac{dz'}{2\pi i}\frac{1}{(z-z')^2}e^{-n\zeta_r(z)-m\zeta_s(z')}, \quad n, m \geq 1.$$

$$\begin{aligned}
\bar{C}^{12}_{11} &= \frac{e^{2\tau_1+i2\sigma_1}}{2}\frac{(1-Z)}{Z^4}, \quad \bar{C}^{13}_{11} = \frac{e^{\tau_1+i\sigma_1-\tau_2-i\sigma_2}}{2}\frac{(1-Z)}{Z}, \\
\bar{C}^{14}_{11} &= \frac{e^{\tau_1+i\sigma_1-\tau_2-i\sigma_2}}{2}\frac{1}{Z}, \quad \bar{C}^{23}_{11} = \frac{e^{\tau_1+i\sigma_1-\tau_2-i\sigma_2}}{2}\frac{1}{Z}, \\
\bar{C}^{24}_{11} &= \frac{e^{\tau_1+i\sigma_1-\tau_2-i\sigma_2}}{2}\frac{(1-Z)}{Z}. \quad \bar{C}^{34}_{11} = \frac{e^{-2\tau_2-i2\sigma_2}}{2}(1-Z).
\end{aligned}$$

Scattering amplitude for four gravitons

$$
\begin{aligned}
\mathcal{A}_{[4]} \quad = \quad & g^2 \int |Z_4|^4 d^2 Z \prod_{r<s} |Z_r - Z_s|^{2\, \frac{p^{(r)}}{2} \cdot \frac{p^{(s)}}{2}}\, e^{-2 \sum_{r=1}^{4} \bar{C}^{[4]rr}_{00}} \\
& \langle 0 | \left\{ \prod_{i=1}^{4} h_{\mu\nu}(p^{(i)}) a_1^{(i)\mu} \tilde{a}_1^{(i)\nu} \right\} \frac{1}{2!} \cdot \frac{1}{2^4} \Bigg\{ \left(\sum_{r,s} \bar{C}^{rs*}_{11} a_1^{(r)\dagger} \cdot a_1^{(s)\dagger} \right)^2 \\
& + \left(\sum_{r,s} \bar{C}^{rs*}_{11} a_1^{(r)\dagger} \cdot a_1^{(s)\dagger} \right) \left(\sum_{r,s} \bar{C}^{rs*}_{10} a_1^{(r)\dagger} \cdot p^{(s)} \right)^2 \Bigg\} \\
& \frac{1}{2!} \cdot \frac{1}{2^4} \Bigg\{ \left(\sum_{r,s} \bar{C}^{rs}_{11}\, \tilde{a}_1^{(r)\dagger} \cdot \tilde{a}_1^{(s)\dagger} \right)^2 + \left(\sum_{r,s} \bar{C}^{rs}_{11}\, \tilde{a}_1^{(r)\dagger} \cdot \tilde{a}_1^{(s)\dagger} \right) \\
& \left(\sum_{r,s} \bar{C}^{rs}_{10} \tilde{a}_1^{(r)\dagger} \cdot p^{(s)} \right)^2 \Bigg\} |0\rangle .
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_{[4]} = \frac{g^2}{2^{10}} \int d^2 Z \, |Z|^{-\frac{s}{4}} |1-Z|^{-\frac{t}{4}} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4} \\
\Bigg\{ \frac{4}{Z^{*2}} \eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + 4\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \frac{4}{(1-Z^*)^2}\, \eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} \\
-\eta^{\mu_1\mu_2} \frac{1}{(1-Z^*)Z^{*2}} \left(Z^* p^{(1)\mu_3} + p^{(4)\mu_3}\right)\left(Z^* p^{(2)\mu_4} + p^{(3)\mu_4}\right) \\
+\eta^{\mu_1\mu_3} \frac{1}{(1-Z^*)Z^{*}} \left(p^{(1)\mu_2} + Z^* p^{(4)\mu_2}\right)\left(Z^* p^{(2)\mu_4} + p^{(3)\mu_4}\right) \\
-\eta^{\mu_1\mu_4} \frac{1}{(1-Z^*)^2 Z^{*}} \left((1-Z^*) p^{(4)\mu_2} + p^{(3)\mu_2}\right)\left((1-Z^*) p^{(1)\mu_3} + p^{(2)\mu_3}\right) \\
-\eta^{\mu_2\mu_3} \frac{1}{(1-Z^*)^2 Z^{*}} \left((1-Z^*) p^{(3)\mu_1} + p^{(4)\mu_1}\right)\left(p^{(1)\mu_4} + (1-Z^*) p^{(2)\mu_4}\right) \\
+\eta^{\mu_2\mu_4} \frac{1}{(1-Z^*)Z^{*}} \left(Z^* p^{(3)\mu_1} + p^{(2)\mu_1}\right)\left(Z^* p^{(1)\mu_3} + p^{(4)\mu_3}\right) \\
-\eta^{\mu_3\mu_4} \frac{1}{(1-Z^*)Z^{*2}} \left(Z^* p^{(3)\mu_1} + p^{(2)\mu_1}\right)\left(p^{(1)\mu_2} + Z^* p^{(4)\mu_2}\right) \Bigg\} \Bigg\{ Z^* \Rightarrow Z
\end{aligned}
$$

Kawai-Lewellen-Tye (KLT) Relations

$$
\begin{aligned}
I &= \int d^2 Z \prod_{r<s} |Z_r - Z_s|^{2\,\frac{p^{(r)}}{2}\cdot\frac{p^{(s)}}{2}} (Z^*)^{-2n} Z^{-2m} (1-Z^*)^{-2p} (1-Z)^{-2q} \\
&= \int d^2 Z \, Z^{-2m-\frac{s}{8}} (1-Z)^{-2q-\frac{t}{8}} (Z^*)^{-2n-\frac{s}{8}} (1-Z^*)^{-2p-\frac{t}{8}}.
\end{aligned}
$$

If we write $Z = x + iy$, then we treat x and y as independent two complex variables. The integrand is an analytic function of y with four branch points, $\pm x$, $\pm i(1-x)$. We may deform the contour of y which is along the real line to the contour along the imaginary axis.

$$
\begin{aligned}
I &= \sin\left(\frac{\pi}{8} t\right) \int_0^1 d\xi \, |\xi|^{-\frac{s}{8}} |1-\xi|^{-\frac{t}{8}} \xi^{-2m} (1-\xi)^{-2q} \\
& \quad \int_1^\infty d\eta \, |\eta|^{-\frac{s}{8}} |1-\eta|^{-\frac{t}{8}} \eta^{-2n} (1-\eta)^{-2p} .
\end{aligned}
$$

Kawai-Lewellen-Tye (KLT) Relations

Factorization

$$\begin{aligned}
I &= \sin\left(\frac{\pi t}{8}\right) I_1(n,p) I_2(m,q), \\
I_1(n,p) &= (-1)^{2p} \frac{\Gamma\left(-\frac{u}{8}+2n+2p-1\right)\Gamma\left(-\frac{t}{8}-2p+1\right)}{\Gamma\left(\frac{s}{8}+2n\right)}, \\
I_2(m,q) &= \frac{\Gamma\left(-\frac{s}{8}-2m+1\right)\Gamma\left(-\frac{t}{8}-2q+1\right)}{\Gamma\left(\frac{u}{8}-2m-2q+2\right)}
\end{aligned}$$

where n, p, m, q are integers or half-integers.

Four-graviton scattering

$$
\begin{aligned}
\mathcal{A}_{[4G]} \;=\;& g^2 c_{[4]} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4} \sin\left(\frac{\pi t}{8}\right) \\
& \Big\{ I_1(2,0)\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + I_1(0,0)\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + I_1(0,2)\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} \\
& -\frac{1}{4}\eta^{\mu_1\mu_2}\left(I_1(0,1)p^{(1)\mu_3}p^{(2)\mu_4} + I_1(1,1)p^{(1)\mu_3}p^{(3)\mu_4} + I_1(1,1)p^{(4)\mu_3}p^{(2)\mu_4} + I_1(2,1)p^{(4)\mu_3}p^{(3)\mu_4}\right) \\
& +\frac{1}{4}\eta^{\mu_1\mu_3}\left(I_1(0,1)p^{(1)\mu_2}p^{(2)\mu_4} + I_1(1,1)p^{(1)\mu_2}p^{(3)\mu_4} + I_1(-1,1)p^{(4)\mu_2}p^{(2)\mu_4} + I_1(0,1)p^{(4)\mu_2}p^{(3)\mu_4}\right) \\
& -\frac{1}{4}\eta^{\mu_1\mu_4}\left(I_1(1,0)p^{(4)\mu_2}p^{(1)\mu_3} + I_1(1,1)p^{(4)\mu_2}p^{(2)\mu_3} + I_1(1,1)p^{(3)\mu_2}p^{(1)\mu_3} + I_1(1,2)p^{(3)\mu_2}p^{(2)\mu_3}\right) \\
& -\frac{1}{4}\eta^{\mu_2\mu_3}\left(I_1(1,0)p^{(3)\mu_1}p^{(2)\mu_4} + I_1(1,1)p^{(3)\mu_1}p^{(1)\mu_4} + I_1(1,1)p^{(4)\mu_1}p^{(2)\mu_4} + I_1(1,2)p^{(4)\mu_1}p^{(1)\mu_4}\right) \\
& +\frac{1}{4}\eta^{\mu_2\mu_4}\left(I_1(-1,1)p^{(3)\mu_1}p^{(1)\mu_3} + I_1(0,1)p^{(3)\mu_1}p^{(4)\mu_3} + I_1(0,1)p^{(2)\mu_1}p^{(1)\mu_3} + I_1(1,1)p^{(2)\mu_1}p^{(4)\mu_3}\right) \\
& -\frac{1}{4}\eta^{\mu_3\mu_4}\left(I_1(0,1)p^{(3)\mu_1}p^{(4)\mu_2} + I_1(1,1)p^{(3)\mu_1}p^{(1)\mu_2} + I_1(1,1)p^{(2)\mu_1}p^{(4)\mu_2} + I_1(2,1)p^{(2)\mu_1}p^{(1)\mu_2}\right)\Big\} \\
& \Big\{ I_1(n,p) \Rightarrow I_2(n,p),\;\; \mu \Rightarrow \nu \Big\}
\end{aligned}
$$

Four-graviton scattering

$$
\begin{aligned}
\mathcal{A}_{[4]} \;=\;& \frac{\kappa^2}{128}\frac{\Gamma\left(-\frac{s}{8}\right)\Gamma\left(-\frac{t}{8}\right)\Gamma\left(-\frac{u}{8}\right)}{\Gamma\left(1+\frac{s}{8}\right)\Gamma\left(1+\frac{t}{8}\right)\Gamma\left(1+\frac{u}{8}\right)} h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4} \\
& \Bigg\{\frac{tu}{2}\frac{1}{s/8+1}\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + \frac{st}{2}\frac{1}{u/8+1}\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \frac{us}{2}\frac{1}{t/8+1}\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} \\
& -\eta^{\mu_1\mu_2}\left(-\frac{s}{2}p^{(1)\mu_3}p^{(2)\mu_4} + \frac{u}{2}p^{(1)\mu_3}p^{(3)\mu_4} + \frac{u}{2}p^{(4)\mu_3}p^{(2)\mu_4} - \frac{u(u-2)}{4}\frac{1}{\frac{s}{8}+1}p^{(4)\mu_3}p^{(3)\mu_4}\right) \\
& +\eta^{\mu_1\mu_3}\left(-\frac{s}{2}p^{(1)\mu_2}p^{(2)\mu_4} + \frac{u}{2}p^{(1)\mu_2}p^{(3)\mu_4} + \frac{s(s-2)}{4}\frac{1}{\frac{u}{8}+1}p^{(4)\mu_2}p^{(2)\mu_4} - \frac{s}{2}p^{(4)\mu_2}p^{(3)\mu_4}\right) \\
& -\eta^{\mu_1\mu_4}\left(-\frac{t}{2}p^{(4)\mu_2}p^{(1)\mu_3} + \frac{u}{2}p^{(4)\mu_2}p^{(2)\mu_3} + \frac{u}{2}p^{(3)\mu_2}p^{(1)\mu_3} - \frac{u(u-2)}{4}\frac{1}{\frac{t}{8}+1}p^{(3)\mu_2}p^{(2)\mu_3}\right) \\
& -\eta^{\mu_2\mu_3}\left(-\frac{t}{2}p^{(3)\mu_1}p^{(2)\mu_4} + \frac{u}{2}p^{(3)\mu_1}p^{(1)\mu_4} + \frac{u}{2}p^{(4)\mu_1}p^{(2)\mu_4} - \frac{u(u-2)}{4}\frac{1}{\frac{t}{8}+1}p^{(4)\mu_1}p^{(1)\mu_4}\right) \\
& +\eta^{\mu_2\mu_4}\left(\frac{s(s-2)}{4}\frac{1}{\frac{u}{8}+1}p^{(3)\mu_1}p^{(1)\mu_3} - \frac{s}{2}p^{(3)\mu_1}p^{(4)\mu_3} - \frac{s}{2}p^{(2)\mu_1}p^{(1)\mu_3} + \frac{u}{2}p^{(2)\mu_1}p^{(4)\mu_3}\right) \\
& -\eta^{\mu_3\mu_4}\left(-\frac{s}{2}p^{(3)\mu_1}p^{(4)\mu_2} + \frac{u}{2}p^{(3)\mu_1}p^{(1)\mu_2} + \frac{u}{2}p^{(2)\mu_1}p^{(4)\mu_2} - \frac{u(u-2)}{4}\frac{1}{\frac{s}{8}+1}p^{(2)\mu_1}p^{(1)\mu_2}\right)\Bigg\}\Big\{\mu \Rightarrow \nu\Big\}
\end{aligned}
$$

Four-graviton scattering

- The Four-graviton scattering amplitude is completely factorized (KLT promoted to 2nd quantized level).
- Contains tachyon poles which have been missing.
- The first three terms become softer in the high energy limit.
- Can be extended to the scattering amplitude in the off-shell region.
- Test of Double Copy and Double Field Theory.

Using the momentum conservation, on-shell consition $s+t+u=0$, and the gauge condition, in the zero-slope limit

$$
\begin{aligned}
\mathcal{A}_{[4]} &= \frac{g^2\pi}{2^{10}} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4} \frac{1}{stu} \\
& \quad \left\{ \frac{ut}{2}\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + \frac{st}{2}\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \frac{su}{2}\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} \right. \\
& \quad -\eta^{\mu_1\mu_2}\left(tp^{(1)\mu_3}p^{(2)\mu_4} + up^{(2)\mu_3}p^{(1)\mu_4}\right) \\
& \quad -\eta^{\mu_1\mu_3}\left(tp^{(1)\mu_2}p^{(3)\mu_4} + sp^{(3)\mu_2}p^{(1)\mu_4}\right) \\
& \quad -\eta^{\mu_1\mu_4}\left(up^{(1)\mu_2}p^{(4)\mu_3} + sp^{(4)\mu_2}p^{(1)\mu_3}\right) \\
& \quad -\eta^{\mu_2\mu_3}\left(up^{(2)\mu_1}p^{(3)\mu_4} + sp^{(3)\mu_1}p^{(2)\mu_4}\right) \\
& \quad -\eta^{\mu_2\mu_4}\left(sp^{(4)\mu_1}p^{(2)\mu_3} + tp^{(2)\mu_1}p^{(4)\mu_3}\right) \\
& \quad \left. -\eta^{\mu_3\mu_4}\left(tp^{(3)\mu_1}p^{(4)\mu_2} + up^{(4)\mu_1}p^{(3)\mu_2}\right) \right\} \left\{ \mu \Rightarrow \nu \right\}.
\end{aligned}
$$

Four-Graviton Scattering Amp by using vertex operators

$$
\begin{aligned}
\mathcal{A}_{[4]} = & \frac{\kappa^2}{128} \frac{\Gamma\left(-\frac{s}{8}\right)\Gamma\left(-\frac{t}{8}\right)\Gamma\left(-\frac{u}{8}\right)}{\Gamma\left(1+\frac{s}{8}\right)\Gamma\left(1+\frac{t}{8}\right)\Gamma\left(1+\frac{u}{8}\right)} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4} \\
& \Bigg\{ \frac{ut}{2}\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + \frac{st}{2}\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \frac{su}{2}\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} \\
& -\eta^{\mu_1\mu_2}\left(tp^{(1)\mu_3}p^{(2)\mu_4} + up^{(2)\mu_3}p^{(1)\mu_4}\right) - \eta^{\mu_1\mu_3}\Big(tp^{(1)\mu_2}p^{(3)\mu_4} \\
& +sp^{(3)\mu_2}p^{(1)\mu_4}\Big) - \eta^{\mu_1\mu_4}\left(up^{(1)\mu_2}p^{(4)\mu_3} + sp^{(4)\mu_2}p^{(1)\mu_3}\right) \\
& -\eta^{\mu_2\mu_3}\left(up^{(2)\mu_1}p^{(3)\mu_4} + sp^{(3)\mu_1}p^{(2)\mu_4}\right) \\
& -\eta^{\mu_2\mu_4}\left(sp^{(4)\mu_1}p^{(2)\mu_3} + tp^{(2)\mu_1}p^{(4)\mu_3}\right) \\
& -\eta^{\mu_3\mu_4}\left(tp^{(3)\mu_1}p^{(4)\mu_2} + up^{(4)\mu_1}p^{(3)\mu_2}\right) \Bigg\} \Big\{ \mu \Rightarrow \nu \Big\}.
\end{aligned}
$$

Four-Graviton Scattering Amplitudes

1. B. S. DeWitt, PRD **162**, 1239 (1967): Calculated the four-graviton scattering amplitude by perturbatively expanding the Einstein GR.
2. J. Schwarz, Phys. Rep. **89**, 223 (1982): Four-graviton scattering amplitude in the type-II super-string theory (by using graviton vertex operator).
3. S. Sannan, PRD **34**, 1749 (1986): Equivalence between the DeWitt's four-graviton scattering amplitude and that of string theory calculation.

Four-Graviton Scattering Amplitude in the perturbative Einstein GR

B. S. DeWitt, PRD **162**, 1239 (1967)

$$
\begin{aligned}
\mathrm{sym}[& -\tfrac{1}{4}P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}\eta_{\rho\lambda})-\tfrac{1}{4}P_{12}(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}\eta_{\rho\lambda})-\tfrac{1}{2}P_6(k_{1\nu}k_{2\mu}\eta_{\alpha\beta}\eta_{\sigma\gamma}\eta_{\rho\lambda}) \\
& +\tfrac{1}{4}P_6(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}\eta_{\rho\lambda})+\tfrac{1}{2}P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\rho}\eta_{\gamma\lambda})+\tfrac{1}{2}P_{12}(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\rho}\eta_{\gamma\lambda}) \\
& +P_6(k_{1\nu}k_{2\mu}\eta_{\alpha\beta}\eta_{\sigma\rho}\eta_{\gamma\lambda})-\tfrac{1}{2}P_6(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\rho}\eta_{\gamma\lambda})+\tfrac{1}{2}P_{24}(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}\eta_{\rho\lambda}) \\
& +\tfrac{1}{2}P_{24}(k_{1\nu}k_{1\beta}\eta_{\mu\sigma}\eta_{\alpha\gamma}\eta_{\rho\lambda})+\tfrac{1}{2}P_{12}(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\rho\lambda})+P_{24}(k_{1\nu}k_{2\sigma}\eta_{\beta\mu}\eta_{\alpha\gamma}\eta_{\rho\lambda}) \\
& -P_{12}(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu}\eta_{\rho\lambda})+P_{12}(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}\eta_{\rho\lambda})+P_{12}(k_{1\nu}k_{1\sigma}\eta_{\beta\gamma}\eta_{\mu\alpha}\eta_{\rho\lambda}) \\
& -P_{24}(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\beta\sigma}\eta_{\gamma\rho}\eta_{\lambda\nu})-2P_{12}(k_{1\nu}k_{1\beta}\eta_{\alpha\sigma}\eta_{\gamma\rho}\eta_{\lambda\mu})-2P_{12}(k_{1\sigma}k_{2\gamma}\eta_{\alpha\rho}\eta_{\lambda\nu}\eta_{\beta\mu}) \\
& -2P_{24}(k_{1\nu}k_{2\sigma}\eta_{\beta\rho}\eta_{\lambda\mu}\eta_{\alpha\gamma})-2P_{12}(k_{1\sigma}k_{2\rho}\eta_{\gamma\nu}\eta_{\beta\mu}\eta_{\alpha\lambda})+2P_6(k_1\cdot k_2\eta_{\alpha\sigma}\eta_{\gamma\nu}\eta_{\beta\rho}\eta_{\lambda\mu}) \\
& -2P_{12}(k_{1\nu}k_{1\sigma}\eta_{\mu\alpha}\eta_{\beta\rho}\eta_{\lambda\gamma})-P_{12}(k_1\cdot k_2\eta_{\mu\sigma}\eta_{\alpha\gamma}\eta_{\nu\rho}\eta_{\beta\lambda})-2P_{12}(k_{1\nu}k_{1\sigma}\eta_{\beta\gamma}\eta_{\mu\rho}\eta_{\alpha\lambda}) \\
& -P_{12}(k_{1\sigma}k_{2\rho}\eta_{\gamma\lambda}\eta_{\mu\nu}\eta_{\alpha\beta})-2P_{24}(k_{1\nu}k_{2\sigma}\eta_{\beta\mu}\eta_{\alpha\rho}\eta_{\lambda\gamma})-2P_{12}(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\rho}\eta_{\lambda\alpha}) \\
& +4P_6(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\rho}\eta_{\lambda\mu})]\ .
\end{aligned}
$$

Classical Solutions with D-Brane Sources and Black p-Branes

Closed string field theory action

$$S = \int \left\{ \langle \Phi | \mathcal{K} \Phi \rangle + \frac{g}{3} \big(\langle \Phi | \Phi \circ \Phi \rangle + \langle \Phi \circ \Phi | \Phi \rangle + \langle \Phi | D_p \rangle \right\}.$$

Classical equation of motion with a D-brane source, J_D

$$\mathcal{K}\Phi + g\Phi \circ \Phi = J_D.$$

Perturbative solutions:

$$\Phi = \frac{1}{\mathcal{K} - i\epsilon} J_D - g \frac{1}{\mathcal{K} - i\epsilon} \left\{ \frac{1}{\mathcal{K} - i\epsilon} J_D \circ \frac{1}{\mathcal{K} - i\epsilon} J_D \right\} + \dots .$$

The zero-th order by P. Di Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda, and R. Russo (1997).

Higher Loop Corrections in QCD

1. M-Loop Diagrams of YM
   $\Leftrightarrow$ M-Loop Open String Diagrams with Vertex Operators
   $\Leftrightarrow$ Tree Diagrams of Closed String with M external states
2. Higher Loop Corrections to Correlators of BMN Operators

Collaborators

1. String scattering amplitudes: S. H. Lai, J. C. Lee and Y. Yang (NCTU, Taiwan)
2. Unitarity of graviton scattering amplitudes and R^2 gravity: T. Inami (SKKU, RIKEN)
3. Classical solutions: Kanghoon Lee (IBS)
4. Numerical study and DMFT : Hoonpyo Lee, H. Y. Park (KNU)