Simple description of generalized electromagnetic and gravitational hopfions

Tomasz Smołka

joint work with Jacek Jezierski

The 5th Conference of the Polish Society on Relativity, 2018

Motivation

In short:

- Electromagnetic (weak gravitational) solutions with non-trivial topological properties.
- Hamiltonian energy for linearized gravity.

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$$f(t,r) := \frac{1}{r^2 - t^2} \tag{1}$$

of the wave equation

$$\Box f(t,r) = -\delta(r,t) \tag{2}$$

- For example, $f(t,r) = \mathbf{E} \cdot \mathbf{r}$.

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- Performing an imaginary time translation $t \to t - \imath$ for $f(t, r)$:

$$\Phi_0 = \frac{1}{(r^2 - (t - \imath)^2)} \tag{3}$$

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$$\Delta r \Delta p \geq 4\hbar \tag{4}$$

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  - Generates electromagnetic hopfions – solutions with non-trivial topological properties .

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Magic field–distorted Culomb solution

- There are known other issues in Physics where such imaginary translation gives a new solution with an interesting structure...

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- Example of simply but highly non-trivial solution of Maxwell equations — limit of Kerr–Newman E-M field[2] for $M \to 0$

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The Janis-Newman (J-N) algorithm

- Originally enables one to obtain Kerr from Schwarzschild metric.

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The Janis-Newman (J-N) algorithm

- Originally enables one to obtain Kerr from Schwarzschild metric.
- The J-N[3] prescription for transforming the tensor structure of metric relies on the Newman–Penrose formalism (null tetrad).
- The result of J-N algorithm is an imaginary coordinate transformation:
  Consider Schwarzschild metric in Eddington-Finkelstein coordinates

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r + r^2 d\Omega^2 \tag{5}$$

Transformation $u = \widetilde{u} + \imath a \cos\theta$, $r = \widetilde{r} - ia\cos\theta$ gives Kerr in Eddington-Finkelstein coordinates.

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History of Complex E-M potentials

- Hertz (1889)
- Whittaker (1903): $(\mathbf{E}, \mathbf{B})$ can be expressed as two scalar potentials (F, G)
- Debye (1909)
- Bateman (1915):

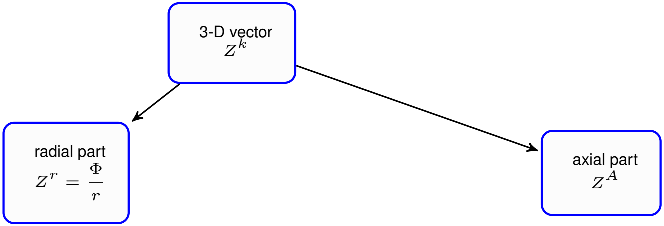
$$\mathbf{Z} = \nabla\xi \times \nabla\eta \tag{6}$$

  is a wave solution of Maxwell equation if

$$\nabla\xi \times \nabla\eta = \imath\left(\partial_t\xi\nabla\eta - \partial_t\eta\nabla\xi\right) \tag{7}$$

- and others...

Hodge-Kodaira decomposition on S^2 for vectors



Z^A can be decomposed into a gradient and co-gradient of some functions

$$Z^A = \alpha^{,A} + \varepsilon^{AB}\beta_{,B} \quad (8)$$

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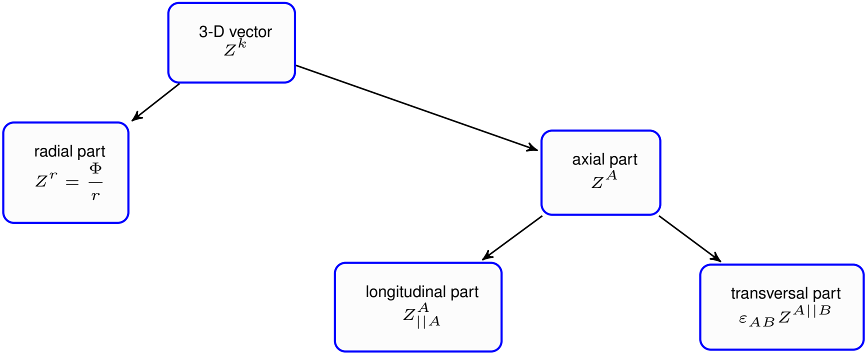

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Quasi-local scalar description

- The following issue will be considered in the spherical coordinates (t, r, θ, ϕ).
- Constructing a scalar $\Phi = \mathbf{Z} \cdot \mathbf{r}$, we receive a complex function which is harmonic

$$\Box\Phi = 0 \tag{9}$$

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- Vacuum Maxwell equations in terms of the scalar Φ take the form

$$\begin{aligned} \partial_r (r\Phi) &= -r^2 Z^A{}_{||A} \\ \partial_t \Phi &= \imath \varepsilon^{AB} Z_{A||B} \\ \Phi &= r Z^r \end{aligned}$$

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- For $Z^A = \alpha^{,A} + \varepsilon^{AB}\beta_{,B}$, we obtain relation between $\boldsymbol{\Delta}\alpha$, $\boldsymbol{\Delta}\beta$ and $\Phi, \partial_t \Phi$...

Generating hopfions

Consider l-th order differential operator A_l which

- generates a function Φ_l from

$$\Phi_0 = \frac{1}{(r^2 - (t - \imath)^2)}$$

  which is proportional to l-th spherical mode.

- commutes with d'Alembert operator.

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Example: For $l = 1$

$$A_1 = \frac{\partial}{\partial x} \pm \imath \frac{\partial}{\partial y} \tag{10}$$

and

$$\begin{aligned} 0 &= A_1 \Box \Phi_0(t, r) \\ &= \Box A_1 \Phi_0(t, r) \\ &= \Box \Phi_1 \end{aligned} \tag{11}$$

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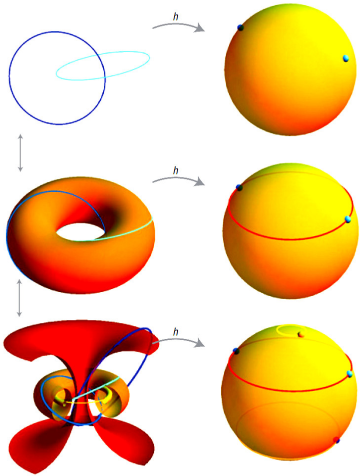
Now, consider topological properties of Φ_1...

Hopf fibration

Hopf fibration is a non-trivial principal bundle of a three-dimensional sphere:

$$S^3 \sim \mathbb{R}^3 \cup \{\infty\} \xrightarrow{h} \mathbb{C} \cup \{\infty\} \sim S^2 \quad (12)$$



W. T. Irvine, D. Bouwmeester *Linked and knotted beams of light* Nature Physics, **4** 716-720, (2008).

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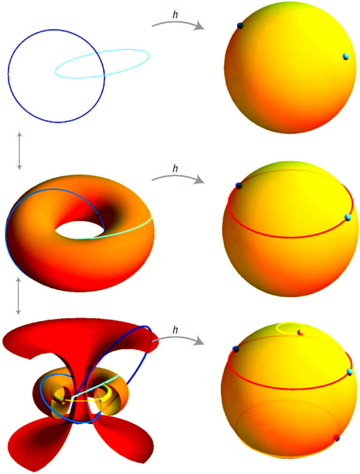

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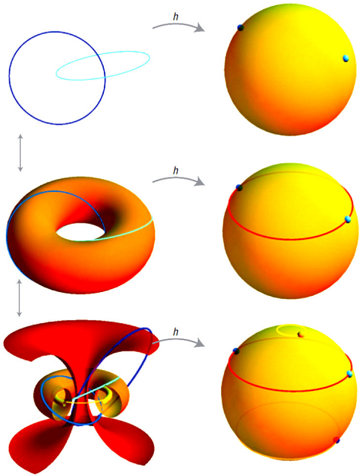
The mapping $h : \mathbb{R}^3 \cup \{\infty\} \xrightarrow{h} \mathbb{C} \cup \{\infty\}$ is given by:

$$h(x, y, z) = \frac{x + \imath z}{-y + \imath\left(\tilde{A} - 1\right)} \quad (13)$$

where $\tilde{A} = \frac{1}{2}(x^2 + y^2 + z^2 + 1)$.



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E-M solution related to Hopf fibration

In 1990, Rañada proposed the following solution of Maxwell equations:

$$\mathbf{E}_R = \frac{1}{4\pi}\frac{\nabla\xi \times \nabla\bar{\xi}}{(1+\xi\bar{\xi})^2} \qquad (14) \qquad\qquad \mathbf{B}_R = \frac{1}{4\pi}\frac{\nabla\eta \times \nabla\bar{\eta}}{(1+\eta\bar{\eta})^2} \qquad (15)$$

where $\xi(t,x,y,z)$ and $\eta(t,x,y,z)$ are defined as

$$\xi = \frac{(Ax+ty)+\imath(Az+t(A-1))}{(tx-Ay)+\imath(A(A-1)-tz)} \qquad (16) \qquad \eta = \frac{(Az+t(A-1))+\imath(tx-Ay)}{(ty+Ax)+\imath(A(A-1)-tz)} \qquad (17)$$

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- The $\mathbf{E}$ field is tangential to lines of constant ξ.
- The helicities are preserved in time.

Topological invariants in physics

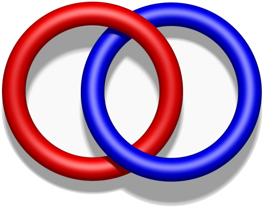
A topologically invariant configuration of field lines is to be linked and/or knotted.

A measure of linkedness is Gauss linking integral:

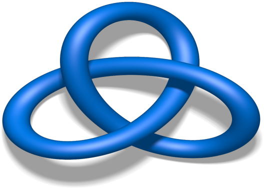
$$L(\mathbf{c}_1, \mathbf{c}_2) = \frac{1}{4\pi} \int \frac{\mathrm{d}\mathbf{c}_1}{\mathrm{d}t_1} \cdot \frac{\mathbf{c}_1 - \mathbf{c}_2}{\|\mathbf{c}_1 - \mathbf{c}_2\|^3} \times \frac{\mathrm{d}\mathbf{c}_2}{\mathrm{d}t_2} \quad (18)$$

where $\mathbf{c}_1(t_1), \mathbf{c}_2(t_2)$ are closed curves.

The self-linking number, $L(\mathbf{c}, \mathbf{c})$, is a measure of knotedness.



Linked, L=1.



Knotted

Topological invariant in physics

Physical analogue of Linking number are helicities[4]. The magnetic helicity:

$$h_m = \int_V dV\, \mathbf{A} \cdot \mathbf{B} \tag{19}$$

where $\mathbf{B} = \mathrm{rot}\,\mathbf{A}$.

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h_m is an „average” of the linking integral over all field-line pairs together, including self-linking.

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h_m is proportional to the well-defined quantity which arises from the algebraical topology. The invariant is called the **Hopf index**. Analogically we can define electric helicity

$$h_e = \int_V dV\,\mathbf{C}\cdot\mathbf{E} \tag{20}$$

where $\mathbf{E} = \mathrm{rot}\,\mathbf{C}$.

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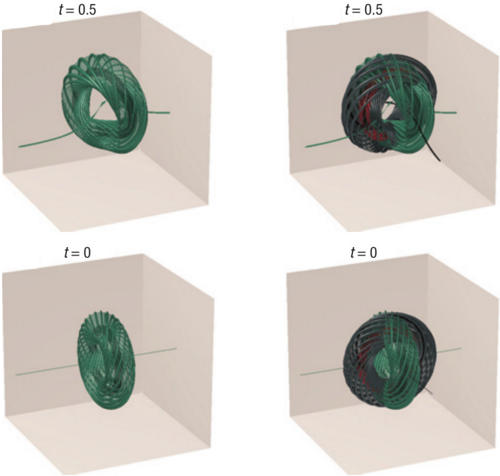
When the E-M field is topologically invariant?

- We assume the fields $\mathbf{Z}, \mathbf{W} = C + \imath A$ are asymptotically decreasing, such that the boundary terms can be neglected.
- The helicities h_m and h_e are time independent if and only if

$$\Im \int_V \mathbf{Z} \cdot \mathbf{Z} = \int_V \mathbf{E} \cdot \mathbf{B} = 0 \tag{21}$$

E-M solution related to Hopf fibration



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- Generalization of Rañada's solutions using Penrose spinorial transform – Bouwmeester group (2008-)

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Linearized theory of gravitation

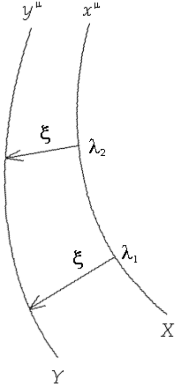
Consider two observers separated by a small spatial vector ξ.
The geodesic deviation equations are

- for a relative tidal acceleration a_i

$$\Delta a_i = -E_{ij}\xi^j \tag{22}$$

- a gyroscope at the tip of ξ will precess with angular velocity Ω_k

$$\Delta\Omega_k = B_{kl}\xi^l \tag{23}$$



Quasi-local scalar description of linearized gravity

3-D symmetric traceless tensor
$Z^{kl} = E^{kl} + \imath B^{kl}$

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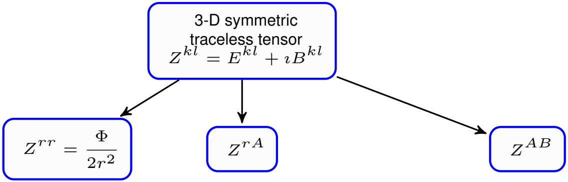

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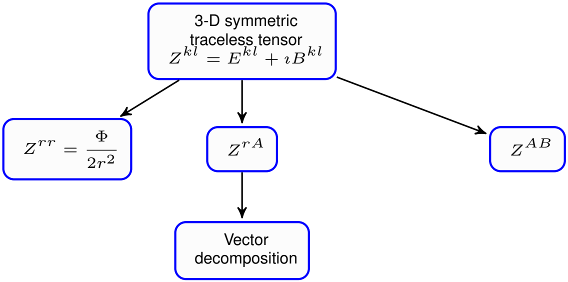

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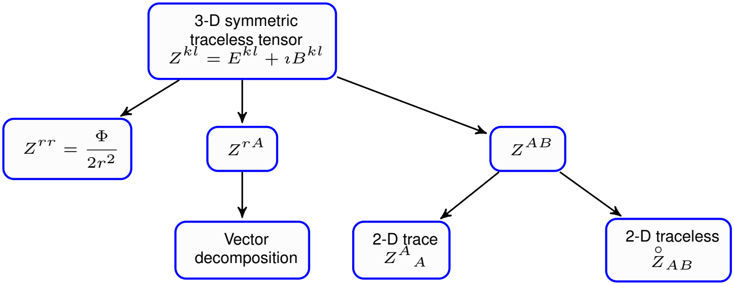

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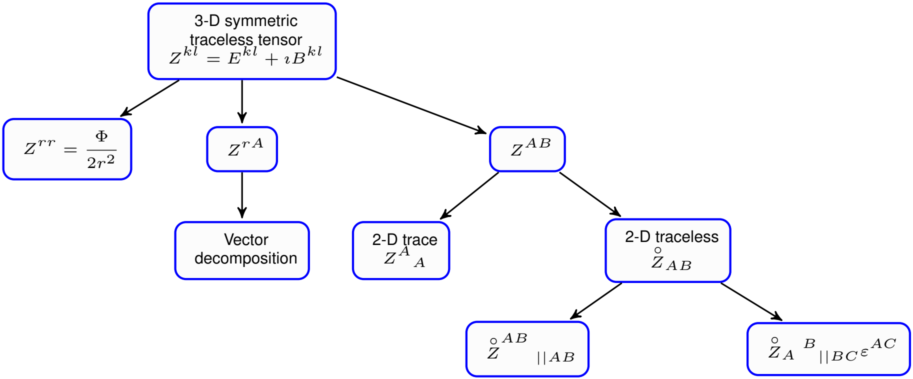

Quasi-local scalar description of linearized gravity

- Introducing a scalar

$$\Psi = 2Z_{kl}x^k x^l = 2Z_{rr}r^2 \tag{24}$$

we receive a complex function which satisfies

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- (2+1) splitting of constraint and dynamical equations enables one to express gravitoelectric tensor components in terms of Ψ and its derivatives.
- Ψ carries gauge independent information of the field.

Quasi-local scalar description of linearized gravity

Constraints:

$$Z^{kl}{}_{|l} = 0 \qquad (26)$$

Dynamical equations:

$$\dot{Z}^{kl} = -\imath \varepsilon^{pq(k} Z^{l)}{}_{q|p} \qquad (27)$$

$$\begin{aligned}
r^2 Z^{rr} &= \frac{1}{2}\Psi \\
r^2 Z_{rA||B}\varepsilon^{rAB} &= -\frac{1}{2}\imath\partial_t\Psi \\
r^3 Z^{rA}{}_{||A} &= -\frac{1}{2}\partial_r(r\Psi) \\
r^2 \overset{(2)}{Z} &= -\frac{1}{2}\Psi \\
r^4 \mathring{Z}^{AB}{}_{||AB} &= \frac{1}{2}\partial_r\left(r\partial_r(r\Psi)\right) \\
r^4 \mathring{Z}_A{}^B{}_{||BC}\varepsilon^{rAC} &= \frac{1}{2}\imath\partial_r(r^2\partial_t\Psi)
\end{aligned}$$

where $\overset{(2)}{Z} = g_{AB}Z^{AB}$ and $\mathring{Z}_{AB} = Z_{AB} - \frac{1}{2}g_{AB}\overset{(2)}{Z}$.

Hopf solution in linearized gravity

The quadrupole solution related to the Hopf fibration

$$\Psi_q := \frac{(x + \imath y)^2}{\left[r^2 - (t - \imath)^2\right]^3} \tag{34}$$

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But it can be easily presented using eigenvectors:

- Each symmetric, traceless, rank 2 tensor can be equivalently represented by its eigenvalues and eigenvectors.
- Both fields E_{kl} and B_{kl} has the same eigenvalues $\{0, \lambda, -\lambda\}$ where

$$\lambda = \frac{[1+x^2+(y+t)^2+z^2]^2}{[1+2(t^2+r^2)+(t^2-r^2)^2]^{5/2}} \tag{35}$$

Hopf solution in linearized gravity

Eigenvalues $\{0, \lambda, -\lambda\}$
Constructing complex vectors for the remaining fields

$$\mathbf{Z_{GE}} = \mathbf{E}_- + i\mathbf{E}_+, \tag{36}$$

$$\mathbf{Z_{GB}} = \mathbf{B}_- + i\mathbf{B}_+, \tag{37}$$

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it turns out

$$\mathbf{Z_{GE}} = e^{i\pi/4}\mathbf{Z_{GB}} \tag{38}$$
$$= e^{iArg(\sqrt{f})}\mathbf{Z_R}, \tag{39}$$

where $f = -(t-i)^2 + r^2$ and $\mathbf{Z_R}$ is the dipole E-M hopfion (Rañada's solution).

Issue of energy for linearized gravity

- The energy related to the Hamiltonian methods is

$$H = \frac{1}{16\pi} \int_\Sigma \mathrm{d}V \left[E_{kl}(-\triangle)^{-1} E^{kl} + B_{kl}(-\triangle)^{-1} B^{kl} \right] \tag{40}$$

[6] I. Białynicki-Birula, *Quantum fluctuations of geometry in a hot Universe, CQG* **32** *(2015) 215015.*

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- In terms of Ψ:

$$\begin{aligned} H \quad = \quad & \frac{1}{32\pi}\int_\Sigma \Big[(r\partial_t\Psi)\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(r\partial_t\overline{\Psi}) \\ & + (r\Psi)_{,r}\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(r\overline{\Psi})_{,r} - \Psi(\boldsymbol{\Delta}+2)^{-1}\overline{\Psi}\Big]\mathrm{d}\Sigma \end{aligned} \tag{41}$$

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- Białynicki–Birula has obtained the same result as a classical limit for Wigner functional[6].

[6] I. Białynicki-Birula, *Quantum fluctuations of geometry in a hot Universe, CQG* **32** *(2015) 215015.*

Proposition of topological charge for linearized gravity

Consider the following non-local objects:

$$h_{GE} = \int_\Sigma E^{ab}(-\triangle^{-1})S_{ab} = \iint_{\Sigma\times\Sigma} \frac{E^{ab}(\mathbf{r}')\,S_{ab}(\mathbf{r}'')}{4\pi\|\mathbf{r}'-\mathbf{r}''\|}\mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r}'' \quad (42)$$

$$h_{GB} = \int_\Sigma B^{ab}(-\triangle^{-1})P_{ab} = \iint_{\Sigma\times\Sigma} \frac{B^{ab}(\mathbf{r}')\,P_{ab}(\mathbf{r}'')}{4\pi\|\mathbf{r}'-\mathbf{r}''\|}\mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r}'' \quad (43)$$

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h_{GB} &= \int_{\Sigma} B^{ab}(-\triangle^{-1})P_{ab} = \iint_{\Sigma\times\Sigma} \frac{B^{ab}\left(\mathbf{r}'\right) P_{ab}\left(\mathbf{r}''\right)}{4\pi\|\mathbf{r}'-\mathbf{r}''\|}\mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r}'' && (43)
\end{aligned}
$$

Conserved in time if

$$
\Im \int_{\Sigma} (E^{kl} + \imath B^{kl})(-\triangle^{-1})(E_{kl} + \imath B_{kl}) = 0 \qquad (44)
$$

Literature

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