For the next Galactic event (several/century..), CCSN GW signatures should be clarified in advance: Hydrodynamics modeling of exploding stars!

1\textsuperscript{st} . General Introduction
✓ Why multi-messengers (inc. GW)?
✓ Basics of GW Physics and Detection
✓ First detection of GW150914

2\textsuperscript{nd} . Core-collapse supernova theory: how to solve “\textit{numerically}” the space-time evolution of dying stars (40 min)

3\textsuperscript{rd} . GW signatures from core-collapse supernovae: what we can learn from future GW observation? (60 min)
Standard scenario of core-collapse SNe

(e.g., Kotake+06, Foglizzo+14, Mezzacappa+15, Janka17 for a review)

core collapse

\[ M \gtrsim 8 M_\odot \]
\[ e^- + p \rightarrow \nu_e + n \]
\[ Fe + \gamma \rightarrow p + n \]

\[ v \]

\[ \rho_e \sim 10^{12} \text{g/cm}^3 \]

\[ v \]

\[ \rho_e \sim 3 \times 10^{14} \text{g/cm}^3 \]

SN explosion

\[ E_{\text{exp}} \sim 10^{51} \text{erg} \]

shock in envelope

shock propagation in core
Standard scenario of core-collapse SNe
(e.g., Kotake+06, Foglizzo+14, Mezzacappa+15, Janka17 for a review)

- Have to find the way how to revive the stalled bounce shock!
- The best-studied & most promising: the neutrino-heating mechanism (Bethe, Wilson 1985)
  : neutrinos heat material to produce explosions!

**The Neutrino-heating mechanism** (Bethe, Wilson, Mayle 1985)
## Two candidate mechanisms of core-collapse supernovae

*(Lecture by T. Foglizzo, reviews in Janka ('17), Müller ('16), Foglizzo+('15), Burrows('13), Kotake+ ('12))*

<table>
<thead>
<tr>
<th><strong>Neutrino mechanism</strong></th>
<th><strong>MHD mechanism</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Progenitor</td>
<td>Rapidly rotation with strong B</td>
</tr>
<tr>
<td>Non- or slowing- rotating star</td>
<td>($\Omega_0 &gt; \sim \pi \text{ rad/s}, B_0 &gt; \sim 10^{11} \text{ G}$)</td>
</tr>
<tr>
<td>Key ingredients</td>
<td><strong>✓ Field winding and the MRI</strong></td>
</tr>
<tr>
<td>✓ Turbulent Convection and SASI</td>
<td>(e.g., Obergaulinger &amp; Aloy (2017), Rembiasz et al. (2016), Moesta et al. (2016), Masada + (2015))</td>
</tr>
<tr>
<td>✓ Precollapse Inhomogenities/structures</td>
<td>✓ Non-Axisymmetric instabilities</td>
</tr>
<tr>
<td>(e.g., B.Mueller et al. (17), Suwa &amp; Mueller (16))</td>
<td>(e.g., Takiwaki, et al. (2016), Summa et al. (2017))</td>
</tr>
<tr>
<td>✓ Novel microphysics: Bollig+(17), Fischer+(18)</td>
<td>✓ Novelties:</td>
</tr>
<tr>
<td>Progenitor fraction</td>
<td>Main players</td>
</tr>
<tr>
<td>Main players</td>
<td>~&lt;1% (Woosley &amp; Heger (07), ApJ):</td>
</tr>
<tr>
<td>~&lt;1% (hypothetical link to magnetar, collapsar)</td>
<td></td>
</tr>
</tbody>
</table>

### Key Considerations:

- **20 M\(_{\odot}\)** from Melson et al. ('16)
- **11.2 M\(_{\odot}\)** from Nakamura et al. in prep.
- **15 M\(_{\odot}\)** star from Lentz et al. ('15)

(see also, Burrows et al. ('17), Melson et al. ('15), Lentz et al. ('15), Roberts et al. ('16), B. Mueller ('15), Takiwaki et al. ('16))
Requirements of CCSN simulations

Ultimate goal:
7D Boltzmann transport in full GR Magneto-hydrodynamics (MHD) with accurate microphysical inputs

Disclaimer: only CCSNs

Code development

General relativity
Quick review; how to evolve hydrodynamics equations (1/3)

Density, velocity, temperature $(x_i, y_i, z_i)$ .... on the grid

Hydrodynamics equations: Non-linear $\Rightarrow$ Computational Fluid Dynamics (CFD)
Quick review; how to evolve hydrodynamics equations (1/3)

Closed set of hydro equations:

<table>
<thead>
<tr>
<th>Equation</th>
<th>Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass conservation: [ \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0 ]</td>
<td>(\rho)</td>
</tr>
<tr>
<td>Momentum conservation: [ \frac{\partial \rho v_i}{\partial t} + \frac{\partial \pi_{ij}}{\partial x_j} = \rho g_i ]</td>
<td>(\mathbf{v})</td>
</tr>
<tr>
<td>Energy conservation: [ \frac{\partial (\rho e)}{\partial t} + \text{div}[(\rho e + p) \mathbf{v}] = -\rho \mathbf{v} \cdot \nabla \Phi ]</td>
<td>(p)</td>
</tr>
<tr>
<td>Equation of State: EOS: (P(\rho, T, Y_e))</td>
<td>(g_i)</td>
</tr>
<tr>
<td>Poisson eq.: (\Delta \Phi = 4\pi G \rho)</td>
<td>(\Phi)</td>
</tr>
</tbody>
</table>

Hydrodynamics equations: Non-linear ⇒ Computational Fluid Dynamics (CFD)
CFD: in essence.. The Riemann problem

\[
\begin{array}{c}
\rho_L \\
u_L \\
P_L \\
\rho_R \\
u_R \\
P_R \\
\end{array}
\]

\[
\begin{array}{c}
\rho_1 \\
u_1 = 0 \\
x_D \\
p_5 \\
u_5 = 0 \\
x \\
\end{array}
\]
CFD: in essence.. The Riemann problem

Riemann Problem:

Reverse shock (RS)

Forward shock (FS)

Pressure

Velocity

Density

head of rarefaction
tail
contact discontinuity

x

x_D
Quick review; how to evolve hydrodynamics equations (3/3)

\[
U_t + F(U)_x = 0
\]

Discretization

\[
\frac{U^{n+1}_i - U^n_i}{\Delta t} = -\frac{F^n_{i-1/2} - F^n_{i+1/2}}{\Delta x}
\]

Solve the Riemann problem

- Godunov method (Godunov, 1959)
- Roe (Roe, 1981)
- HLLD · C (Toro et al. 1994)
- HLL (Einfeldt, 1988)
3D Newtonian simulations of rapidly rotating core-collapse of a 27 $M_{\odot}$ star ($\Omega_0 = 2$ rad/s) (from Takiwaki & Kotake, 2018, MNRAS Letters)
Why GR ? Introduction to Numerical Relativity

Given the stellar core mass \( M \), if the core radius \( R \) is smaller than Schwarzschild radius, a Black hole (BH) forms.

\[
\text{Schwarzschild radius } \quad r_s = \frac{2GM}{c^2} = 2.95\left(\frac{M}{M_\odot}\right) \text{ km}
\]

\( (P)\text{NS radius } \approx R \approx 10-50\text{ km} \)

\( \frac{r_s}{R} \approx 10\%: \text{GR effect is significant in the stellar center!} \)

See textbooks by Baumgarte and Shapiro, Shibata, Rezzolla

Stating point;
Solving dynamics of space-time (1/4)

For a flat spacetime: Minkowski

$$\eta_{\mu\nu} \equiv \text{diag}(-1, 1, 1, 1)$$

Flat space-time: \(d^2s = \eta_{\mu\nu}dx^\mu dx^\nu\)

Minkowski

Curved spacetime

In the curved spacetime,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

\(g_{\mu\nu}(x^\alpha)\) depends on space-time

Pythagorean theorem

$$d^2s = \eta_{\mu\nu}dx^\mu dx^\nu = -d\bar{t}^2 + \sum (dx^i)^2$$

\(dx^0 = dt, dx^i = dx\)

\(d\bar{t} = \alpha dt\), \(\alpha\): lapse

\(d\bar{x}^i = dx^i + \beta^i dt\), \(\beta^i\): shift vector

Need to determine 10 (variables or d.o.f.)

Need “Numerical relativity”.

(Analytic solution only in special cases, Schwarzschild, Kerr)
Solving dynamics of space-time (2/4)

Again, the Minkowski metric.

\[ g^{\mu \nu} \rightarrow \eta^{\mu \nu} \equiv \text{diag}(-1, 1, 1, 1) \]

Need to solve Einstein equation

\[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \frac{8 \pi G}{c^4} T_{\mu \nu} \]

In the limit of \( \frac{R_s}{R} \to 0 \)

\[ \nabla^2 \phi = 4 \pi G \rho \]

\( R_{\mu \nu} \) Ricci tensor

Energy-momentum tensor

\[ T^{\mu \nu} = (e + p) u^\mu u^\nu + p g^{\mu \nu} \]

\( R_{\mu \nu}, R \) are the functional of \( g_{\mu \nu} \)

\[ R_{\mu \nu} = \Gamma^\alpha_{\mu \nu, \alpha} - \Gamma^\alpha_{\mu \alpha, \nu} + \Gamma^\alpha_{\gamma \alpha} \Gamma^\gamma_{\mu \nu} - \Gamma^\alpha_{\gamma \nu} \Gamma^\gamma_{\mu \alpha} \]

\[ \Gamma^\alpha_{\beta \gamma} \equiv \frac{1}{2} g^\alpha_{\mu \nu} \left( g_{\mu \beta, \gamma} + g_{\mu \gamma, \beta} - g_{\beta \gamma, \mu} \right) \]

Ricci scalar
Solving dynamics of space-time (3/4)

✓ 3+1 decomposition (see textbooks by E. Gourgoulhon, M. Shibata, L. Rezzolla..)

1. Set space-like hyper surface at \( t \)

2. Time-like unit vector \( \mathbf{n} \) defined

3. Evolve \( \gamma_{ij} \), \& \( K_{ij} \) (\( \propto \dot{\gamma}_{ij} \))

\[
\sum_{t+\delta t} \Delta \rightarrow \sum_{t}
\]

\( \partial_t \gamma_{ij} \), \& \( K_{ij} \) (\( \propto \dot{\gamma}_{ij} \))

Analogy with EM

\[
\begin{align*}
G_{\mu\nu} n^\mu n^\nu &= 8\pi T_{\mu\nu} n^\mu n^\nu : \text{Hamiltonian constraint} \\
G_{\mu\nu} \gamma_k^\nu &= 8\pi T_{\mu\nu} \gamma_k^\nu : \text{Momentum constraint} \\
G_{\mu\nu} \gamma_i^\mu \gamma_j^\nu &= 8\pi T_{\mu\nu} \gamma_i^\mu \gamma_j^\nu : \text{Evolution equation}
\end{align*}
\]

\( \gamma_{ij} \): 3-metric, \( n^\mu \): timelike normal

Initial Einstein eqns in 3+1 formalism

\[
\begin{align*}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i \\
\partial_t K_{ij} &= -\nabla_i \nabla_j \alpha + \alpha (R_{ij} + K K_{ij} - 2 K_{ik} K_{kj}) \\
&\quad + \beta^k \nabla_k K_{ij} + K_{ik} \nabla_j \beta^k + K_{jk} \nabla_i \beta^k \\
&\quad - 8\pi \alpha (S_{ij} - \gamma_{ij}^e (S - \rho_H)) \\
0 &= R + K^2 - K_{ij} K^{ij} - 16\pi \rho_H \\
0 &= \nabla_i (K^{ij} - \gamma^{ij} K) - 8\pi S^j
\end{align*}
\]

Degree of freedom = 10

- 4 (Hamiltonian 1, Momentum 3)
- 4 (\( \alpha, \beta^1 \))
- 2 (GW: +, x mode)
Solving dynamics of space-time (4/4)

✓ With \( g_{\mu \nu} \), one can solve the (radiation-)hydrodynamics equations!

\[
\begin{align*}
\frac{\partial \sqrt{\gamma} \rho W}{\partial t} + \frac{\partial \sqrt{-g} \rho W v^i}{\partial x^i} & = 0 \\
\frac{\partial \sqrt{\gamma} \rho h W^2 v_j}{\partial t} + \frac{\partial \sqrt{-g} (\rho h W^2 v_j v^i + \delta_j^i P)}{\partial x^i} & = \frac{1}{2} \sqrt{-g} T_{\mu \nu} \frac{\partial g_{\mu \nu}}{\partial x^j} - (\nabla_\beta T_{(\nu)}^{\beta \mu})^c_{\mu} \\
\frac{\partial \sqrt{\gamma} \tau}{\partial t} + \frac{\partial \sqrt{-g} (\tau v^i + P v^i)}{\partial x^i} & = \alpha \sqrt{-g} (T_{\mu 0} \frac{\partial \ln g}{\partial x^\mu} - T^{\mu \nu} T_{\mu 0} T_{0 \nu}) - (\nabla_\beta T_{(\nu)}^{0 \beta})^c_{0}
\end{align*}
\]

Mass cons.
Momentum cons.
Energy cons.

Newtonian limit:
\[
\begin{align*}
D & = \rho W_{\text{Newton}} \rightarrow \rho \\
W & = \frac{1}{\sqrt{1 - v^2}} = 1 + \frac{1}{2} v^2 + \mathcal{O}(v^4) \\
S^i & = \rho h W^2 v^i = \rho (1 + \epsilon + \frac{p}{\rho}) v^i_{\text{Newton}} \rightarrow \rho v^i
\end{align*}
\]

✓ Baumgarte-Shibata-Shapiro-Nakamura (BSSN) formalism: ADM numerically unstable

BSSN variables:
\[
\begin{align*}
\phi & = \frac{1}{12} \ln[\det(\gamma_{ij})] \\
\tilde{\gamma}_{ij} & = e^{-4\phi} \gamma_{ij} \\
K & = \gamma^{ij} K_{ij} \\
\tilde{A}_{ij} & = e^{-4\phi} \left( K_{ij} - \frac{1}{3} \gamma_{ij} K \right) \\
\tilde{\Gamma}^i & = -\tilde{\gamma}^{ij}_{\cdot j} \\
\end{align*}
\]

General Relativistic Simulations
(limit to CCSN context):

✓ AEI-Southampton-Amsterdam
Caltech collaboration:
(e.g., Cactus code
http://cactuscode.org)

✓ Our team:
Kuroda, KK, Takiwaki

✓ Monash-Garching group
(Conformally flatness approximation)
Mueller, Janka et al.
Solving dynamics of neutrino(ν) radiation field (1/3)

✓ Neutrino propagation in supernova core

Neutrino distribution function

✓ β-equilibrium is achieved only in the high-density region!

⇒ Neutrino occupation probability: \( f_\nu \) in the energy space needs to be accurately treated (bottom line).

Levels of approximations:

\[
f(t,r,\theta,\phi,E,\theta_p,\phi_p)
\]

\[
E_R(t,r,\theta,\phi,E) = \int d\theta_p \, d\phi_p \, f
\]

\[
E_R(t,r,\theta,\phi) = \int dE \, d\theta_p \, d\phi_p \, f
\]

“MGMA” (6 dimensional problem)

“MG” (Multi energy-Group) (e.g., MGFLD, M1, IDSA)

“Gray (no energy-dependence)”
Solving dynamics of neutrino($\nu$) radiation field (1/3)

✓ Neutrino propagation in supernova core

Neutrino distribution function

✓ $\beta$-equilibrium is achieved

Levels of approximations:

$$f(t,r,\theta,\phi,E,\theta_p,\phi_p)$$

$$E_R(t,r,\theta,\phi,E) = \int f(t,r,\theta,\phi,E) dE$$

$$E_R(t,r,\theta,\phi) = \int dE f(t,r,\theta,\phi,E)$$

Multi-angle, Multi-energy transport: Ultimate goal!
1. Simple deleptonization (e.g., Liebendoerfer (2005), ApJ)
2. Neutrino leakage scheme (e.g., Kotake et al. (2003), ApJ, Sekiguchi (2009), PTP)
5. Multi-energy flux-limited diffusion, M1 Isotropic Diffusion Source Approximation (e.g., Bruenn (1985), Burrows et al. (2006), Liebendoerfer et al. (2009), Obergaulinger et al. (2016))
7. Full multi-angle Boltzmann transport (e.g., Sumiyoshi and Yamada (2012), Nagakura et al. (2016))

A roadmap how to implement neutrino heating/cooling in your code!

- Easy
  - Neutrino cooling (on)
  - 6 months “only if” your hydro is robust
- Hard
  - Neutrino heating (on)
  - “Manual”
  - > a few years (or a life work!)

Solving dynamics of neutrino(v) radiation field

The Neutrino Boltzmann equation

\[ f(\nu)(t, r, \theta, \phi, E\nu, \theta\nu, \phi\nu) \]

\[
\frac{\partial f(\nu)}{\partial t} + \frac{d\mathbf{r}}{dt} \cdot \frac{\partial f(\nu)}{\partial \mathbf{r}} + \frac{d\mathbf{p}}{dt} \cdot \frac{\partial f(\nu)}{\partial \mathbf{p}} = \frac{df(\nu)}{dt} \bigg|_{\text{coll}}
\]

0th, energy density, \( E(\nu) = E\nu^3 \int d\Omega f(\nu) \hat{\mathbf{p}}_0^0 = E\nu^3 \int d\Omega \)

1st moment, Flux: \( F_\alpha(\nu) = E\nu^3 \int d\Omega f(\nu) \hat{\mathbf{p}}_\alpha^0 \) (note: comoving vs. laboratory frame)

✓ Self-consistent modeling (no parameters for heating/cooling)
✓ Implicit scheme needed!
Current Status of CCSN simulations

Disclaimer: only CCSNs

Ultimate goal:
7D Boltzmann transport in full GR Magneto-hydrodynamics (MHD) with accurate microphysical inputs

Code development

Kuroda et al. (2016,18), ApJ(S)
Müller et al. (2012, 2018)
O'Connor (2015)
Fischer et al. (2014)
Melson et al. (2015)
Hanke et al. (2014)
Sumiyoshi & Yamada (2012)
Nagakura et al. (2014)

General relativity

1D-GR 2D-GR 3D-GR

1D
2D
3D

Transport Dimensions

Multi-Energy

Gray

Adiabatic
General relativistic neutrino transport with detailed v transport: Vertex-CoCoNuT code


✓ Conformal flatness approximation (+)
✓ L.H.S. of Boltzmann eq. is super messy...

\[
\frac{\xi}{\alpha} \left( \frac{\partial f}{\partial t} - \beta^r \frac{\partial f}{\partial r} \right) + \frac{\nu}{\phi^2} \frac{\partial f}{\partial \varphi} - \frac{\varepsilon W^3}{r \alpha \phi^3} \frac{\partial f}{\partial \varphi} \left( \beta^r \phi \left( -\psi - \nu \frac{\partial \nu}{\partial r} \right) + \nu^2 \phi \left( 2r \frac{\partial \phi}{\partial r} - \psi \phi \right) + \right.
\]
\[
\left. r \left( -\mu \frac{\partial \alpha}{\partial r} + \mu^2 \phi^2 \frac{\partial \beta^r}{\partial r} - \frac{\partial \phi^2}{\partial t} \right) \right) + \nu^2 \left[ \mu \frac{\partial \mu}{\partial r} + \phi^2 \left( -\mu \frac{\partial \beta^r}{\partial r} + \frac{\partial \nu}{\partial t} \right) + \right.
\]
\[
\left. r \frac{\partial \phi}{\partial t} - r \beta^r \frac{\partial \phi}{\partial r} \right] + \nu \alpha \left[ \phi \left( \psi - \nu \frac{\partial \nu}{\partial r} \right) + \right.
\]
\[
\left. 2r \psi \frac{\partial \phi}{\partial r} + \phi^2 \left( \mu \frac{\partial \nu}{\partial t} - \beta^r \frac{\partial \nu}{\partial r} \right) + \phi^2 \right] \right) + \frac{W^3 (1 - \mu^2)}{r \alpha \phi^3} \frac{\partial f}{\partial \mu} \left\{ \alpha \left[ \phi \left( \frac{\xi}{W^2} - \nu \frac{\partial \nu}{\partial r} \right) + 2r \xi \frac{\partial \phi}{\partial r} \right] + \right.
\]
\[
\left. \phi \left[ \beta \phi^2 \left( r \xi \frac{\partial \nu}{\partial r} - \frac{\nu}{W^2} \right) - r \frac{\partial \alpha}{\partial t} \left( -\phi^2 \frac{\partial \beta^r}{\partial r} \right) - r \xi \phi^2 \frac{\partial \nu}{\partial r} \right] \right\} = c[f],
\]

Full-3D-GR code with multi-energy neutrino transport (M1)

✓ “FGR”: Fully General Relativistic code with multi-energy neutrino transport


The marriage of BSSNOK formalism (3D GR code, Kuroda & Umeda (2010, ApJS))

+ M1 scheme; Shibata+2011, Thorne 1981, (see also, Just et al. (2015), O'Connor (2015) for recent work)

✓ Evolution equation of neutrino radiation energy

\[
\frac{\partial t \sqrt{\gamma} E(\varepsilon)}{\partial t} + \frac{\partial t \sqrt{\gamma}}{\partial t} (\alpha F^i(\varepsilon) - \beta^i E(\varepsilon)) + \sqrt{\gamma} \alpha \partial_{\varepsilon} (\varepsilon M^{\mu}_{(\varepsilon)} n_{\mu}) = \sqrt{\gamma} (\alpha P^{ij}_{(\varepsilon)} K_{ij} - F^i(\varepsilon) \partial t \alpha - \alpha S^{\mu}_{(\varepsilon)} n_{\mu}),
\]

✓ Evolution equation of radiation flux

\[
\frac{\partial t \sqrt{\gamma} F(\varepsilon)}{\partial t} + \frac{\partial t \sqrt{\gamma}}{\partial t} (\alpha P^{ij}_{(\varepsilon)} F_{ij}(\varepsilon) - \beta^i F(\varepsilon) - \sqrt{\gamma} \alpha \partial_{\varepsilon} (\varepsilon M^{\mu}_{(\varepsilon)} \gamma_{ij\mu}) = \sqrt{\gamma} \left[ -E(\varepsilon) \partial t \alpha + F^i(\varepsilon) \partial t \beta^i + (\alpha / 2) P^{jk}_{(\varepsilon)} \partial t \gamma_{jk} + \alpha S^{\mu}_{(\varepsilon)} \gamma_{ij\mu}\right]
\]

✓ Analytic Closure with the use of Minerbo-type Eddington factor (Murchikova, Abdikamalov + (2017))

\[
p^{ij}_{(\varepsilon)} = \frac{3 \chi_{(\varepsilon)} - 1}{2} p^{ij}_{\text{thin}(\varepsilon)} + \frac{3(1 - \chi_{(\varepsilon)})}{2} p^{ij}_{\text{thick}(\varepsilon)}
\]

\[
\chi_{(\varepsilon)} = \frac{5 + 6 F^2_{(\varepsilon)} - 2 F^3_{(\varepsilon)} + 6 F^4_{(\varepsilon)}}{15}
\]
General relativistic neutrino transport with detailed $\nu$ transport: Vertex-CoCoNuT code


✓ Conformal flatness approximation (+)

✓ L.H.S. of Boltzmann eq. is super messy...

$ds^2 = -c^2 dt^2 + a^2(r) dr^2 + r^2 d\Omega^2$

Full-3D-GR code with multi-energy neutrino transport (M1)

✓ “FGR”: Fully General Relativistic code with multi-energy neutrino transport

The marriage of BSSNOK formalism (3D GR code, Kuroda & Uryū) + M1 scheme; Shibata+2011, Thorne 1981, (see also, Just et al. (2011))

✓ Evolution equation of neutrino radiation energy

$\partial_t \sqrt{\gamma} E_\nu + \partial_i \sqrt{\gamma} (\alpha F^i_{\nu \ell} - \beta E_{\nu \ell}) + \sqrt{\gamma} \alpha \partial_\ell \varepsilon M^\mu (\varepsilon M^\mu_{\nu \ell} n_\mu)$

✓ Analytic Closure with the use of Minerbo-type expressions

$p_{ij} = \frac{3\chi_{\nu \ell}}{2} P_{\text{thin}(\nu \ell)} + \frac{3(1 - \chi_{\nu \ell})}{2} P_{\text{thick}(\nu \ell)}$

✓ Base-line opacity (t.b.updated)
2<sup>nd</sup>. Core-collapse supernova theory: how to solve “numerically” the space-time evolution of dying stars

⇒ Numerical relativity (space-time) + CFD (hydrodynamics) + Neutrino Boltzmann equation (with approximations) self consistently!

3<sup>rd</sup>. GW signatures from core-collapse supernovae: what we can learn from future GW observation?
Useful references

1. Review on Core-Collapse Supernova Theory

2. Books on numerical relativity

3. Books on radiation hydrodynamics