

Quark Pauli blocking in hadronic matter ¹

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Group Seminar "Theory of Elementary Particles"

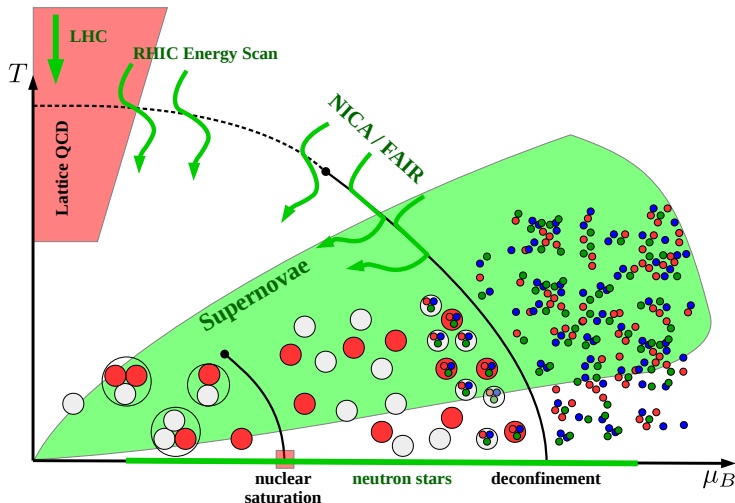
Wrocław, 12 October 2018



¹Collab: N.-U. Bastian, A. Dubinin, A. Friesen, H. Grigorian, G. Röpke, L. Turko

QCD Phase Diagram with Clustering Aspects

From: N.-U. Bastian et al., Universe 4 (2018) 67; arxiv:1804.10178



Φ —Derivable Approach to the Cluster Virial Expansion

$$\Omega = \sum_{l=1}^A \Omega_l = \sum_{l=1}^A \left\{ c_l \left[\text{Tr} \ln (-G_l^{-1}) + \text{Tr} (\Sigma_l G_l) \right] + \sum_{\substack{i,j \\ i+j=l}} \Phi[G_i, G_j, G_{i+j}] \right\} ,$$

$$G_A^{-1} = G_A^{(0)-1} - \Sigma_A , \quad \Sigma_A(1 \dots A, 1' \dots A', z_A) = \frac{\delta \Phi}{\delta G_A(1 \dots A, 1' \dots A', z_A)}$$

Stationarity of the thermodynamical potential is implied

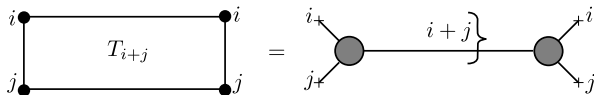
$$\frac{\delta \Omega}{\delta G_A(1 \dots A, 1' \dots A', z_A)} = 0 .$$

Cluster virial expansion follows for this Φ — functional



Figure: The Φ functional for A —particle correlations with bipartitions $A = i + j$.

Green's function and T-matrix: separable approximation



The T_A matrix fulfills the Bethe-Salpeter equation in ladder approximation

$$T_{i+j}(1, 2, \dots, A; 1', 2', \dots, A'; z) = V_{i+j} + V_{i+j} G_{i+j}^{(0)} T_{i+j},$$

which in the separable approximation for the interaction potential,

$$V_{i+j} = \Gamma_{i+j}(1, 2, \dots, i; i+1, i+2, \dots, i+j) \Gamma_{i+j}(1', 2', \dots, i'; (i+1)', (i+2)', \dots, (i+j)'),$$

leads to the closed expression for the T_A matrix

$$T_{i+j}(1, 2, \dots, i+j; 1', 2', \dots, (i+j)'; z) = V_{i+j} \{1 - \Pi_{i+j}\}^{-1},$$

with the generalized polarization function

$$\Pi_{i+j} = \text{Tr} \left\{ \Gamma_{i+j} G_i^{(0)} \Gamma_{i+j} G_j^{(0)} \right\}$$

The one-frequency free i -particle Green's function is defined by the $(i-1)$ -fold Matsubara sum

$$\begin{aligned} G_i^{(0)}(1, 2, \dots, i; \Omega_i) &= \sum_{\omega_1 \dots \omega_{i-1}} \frac{1}{\omega_1 - E(1)} \frac{1}{\omega_2 - E(2)} \cdots \frac{1}{\Omega_i - (\omega_1 + \dots + \omega_{i-1}) - E(i)} \\ &= \frac{(1-f_1)(1-f_2) \dots (1-f_i) - (-)^i f_1 f_2 \dots f_i}{\Omega_i - E(1) - E(2) - \dots - E(i)}. \end{aligned}$$

Useful relationships for many-particle functions

$$G_{i+j}^{(0)} = G_{i+j}^{(0)}(1, 2, \dots, i+j; \Omega_{i+j}) = \sum_{\Omega_i} G_i^{(0)}(1, 2, \dots, i; \Omega_i) G_j^{(0)}(i+1, i+2, \dots, i+j; \Omega_j) .$$

Another set of useful relationships follows from the fact that in the ladder approximation both, the full two-cluster ($i+j$ particle) T matrix and the corresponding Greens' function

$$G_{i+j} = G_{i+j}^{(0)} \{1 - \Pi_{i+j}\}^{-1} \quad (1)$$

have similar analytic properties determined by the $i+j$ cluster polarization loop integral and are related by the identity

$$T_{i+j} G_{i+j}^{(0)} = V_{i+j} G_{i+j} . \quad (2)$$

which is straightforwardly proven by multiplying Equation for the T_{i+j} - matrix with $G_{i+j}^{(0)}$ and using Equation (1). Since these two equivalent expressions in Equation (2) are at the same time equivalent to the two-cluster irreducible Φ functional these functional relations follow

$$\begin{aligned} T_{i+j} &= \delta\Phi / \delta G_{i+j}^{(0)} , \\ V_{i+j} &= \delta\Phi / \delta G_{i+j} . \end{aligned}$$

Next we prove the relationship to the Generalized Beth-Uhlenbeck approach!

GBU EoS from the Φ -derivable approach

Consider the partial density of the A -particle state defined as

$$n_A(T, \mu) = -\frac{\partial \Omega_A}{\partial \mu} = -\frac{\partial}{\partial \mu} d_A \int \frac{d^3 q}{(2\pi)^3} \int \frac{d\omega}{2\pi} \left[\ln(-G_A^{-1}) + \text{Tr}(\Sigma_A G_A) \right] + \sum_{i,j} \Phi[G_i, G_j, G_{i+j}] . \quad (3)$$

Using spectral representation for $F(\omega)$ and Matsubara summation

$$F(iz_n) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\text{Im} F(\omega)}{\omega - iz_n} , \quad \sum_n \frac{c_A}{\omega - iz_n} = f_A(\omega) = \frac{1}{\exp[(\omega - \mu)/T] - (-1)^A}$$

with the relation $\partial f_A(\omega)/\partial \mu = -\partial f_A(\omega)/\partial \omega$ we get for Equation (3) now

$$n_A(T, \mu) = -d_A \int \frac{d^3 q}{(2\pi)^3} \int \frac{d\omega}{2\pi} f_A(\omega) \frac{\partial}{\partial \omega} \left[\text{Im} \ln(-G_A^{-1}) + \text{Im}(\Sigma_A G_A) \right] + \sum_{i,j} \frac{\partial \Phi[G_i, G_j, G_A]}{\partial \mu} ,$$

where a partial integration over ω has been performed For two-loop diagrams of the sunset type holds a cancellation² which we generalize here for cluster states

$$d_A \int \frac{d^3 q}{(2\pi)^3} \int \frac{d\omega}{2\pi} f_A(\omega) \frac{\partial}{\partial \omega} (\text{Re} \Sigma_A \text{Im} G_A) - \sum_{i,j} \frac{\partial \Phi[G_i, G_j, G_A]}{\partial \mu} = 0 .$$

Using generalized optical theorems we can show that ($G_A = |G_A| \exp(i\delta_A)$)

$$\frac{\partial}{\partial \omega} \left[\text{Im} \ln(-G_A^{-1}) + \text{Im} \Sigma_A \text{Re} G_A \right] = 2 \text{Im} \left[G_A \text{Im} \Sigma_A \frac{\partial}{\partial \omega} G_A^* \text{Im} \Sigma_A \right] = -2 \sin^2 \delta_A \frac{\partial \delta_A}{\partial \omega} .$$

The density in the form of a generalized Beth-Uhlenbeck EoS follows

$$n(T, \mu) = \sum_{i=1}^A n_i(T, \mu) = \sum_{i=1}^A d_i \int \frac{d^3 q}{(2\pi)^3} \int \frac{d\omega}{2\pi} f_i(\omega) 2 \sin^2 \delta_i \frac{\partial \delta_i}{\partial \omega} .$$

²B. Vanderheyden & G. Baym, J. Stat. Phys. (1998), J.-P. Blaizot et al., PRD (2001)

Example: Deuterons in Nuclear Matter

The Φ -derivable thermodynamical potential for the nucleon-deuteron system reads

$$\Omega = -\text{Tr} \{ \ln(-G_1) \} - \text{Tr} \{ \Sigma_1 G_1 \} + \text{Tr} \{ \ln(-G_2) \} + \text{Tr} \{ \Sigma_2 G_2 \} + \Phi[G_1, G_2] ,$$

where the full propagators obey the Dyson-Schwinger equations

$$G_1^{-1}(1, z) = z - E_1(p_1) - \Sigma_1(1, z); \quad G_2^{-1}(12, 1'2', z) = z - E_1(p_1) - E_2(p_2) - \Sigma_2(12, 1'2', z),$$

with selfenergies and Φ functional

$$\Sigma_1(1, 1') = \frac{\delta \Phi}{\delta G_1(1, 1')} ; \quad \Sigma_2(12, 1'2', z) = \frac{\delta \Phi}{\delta G_2(12, 1'2', z)} , \quad \Phi =$$


fulfilling stationarity of the thermodynamic potential $\partial\Omega/\partial G_1 = \partial\Omega/\partial G_2 = 0$.

For the density we obtain the cluster virial expansion

$$n = -\frac{1}{V} \frac{\partial \Omega}{\partial \mu} = n_{\text{qu}}(\mu, T) + 2n_{\text{corr}}(\mu, T) ,$$

with the correlation density in the generalized Beth-Uhlenbeck form

$$n_{\text{corr}} = \int \frac{dE}{2\pi} g(E) 2 \sin^2 \delta(E) \frac{d\delta(E)}{dE} .$$

Example: Deuterons in Nuclear Matter

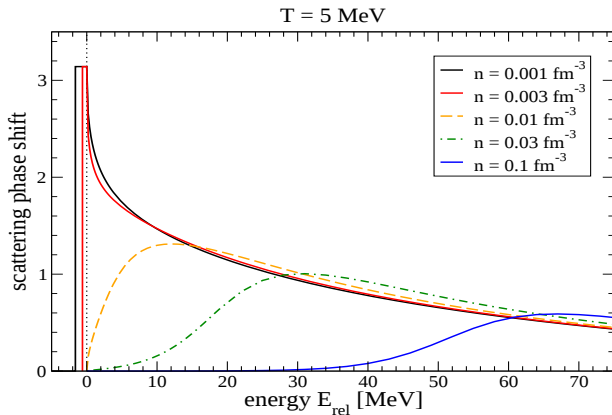


Figure: Integrand of the intrinsic partition function as function of the c.m.s. energy in the deuteron channel. Mott dissociation and Levinson's theorem!
From G. Röpke, J. Phys. Conf. Ser. 569 (2014) 012014.

Cluster Virial Expansion for Quark-Hadron Matter within the Φ Derivable Approach

$$\begin{aligned}\Omega &= \sum_{i=Q,M,D,B} c_i \left[\text{Tr} \ln (-G_i^{-1}) + \text{Tr} (\Sigma_i G_i) \right] + \Phi [G_Q, G_M, G_D, G_B] , \\ &= \sum_{i=Q,M,D,B} d_i \int \frac{d^3 q}{(2\pi)^3} \int \frac{d\omega}{2\pi} \left\{ \omega + 2T \ln \left[1 - e^{-\omega/T} \right] \right\} 2 \sin^2 \delta_i \frac{\partial \delta_i}{\partial \omega} .\end{aligned}$$

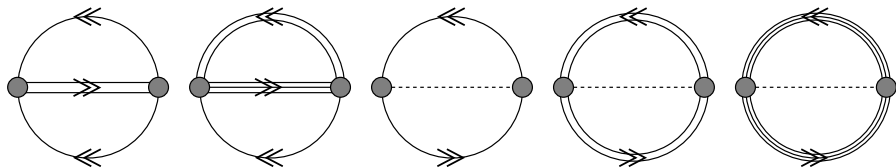


Figure: Φ functional for the quark-meson-diquark-baryon system in 2-loop approx.

$$\Sigma_i = \frac{\delta \Phi [G_Q, G_M, G_D, G_B]}{\delta G_i} .$$

The selfenergies ...

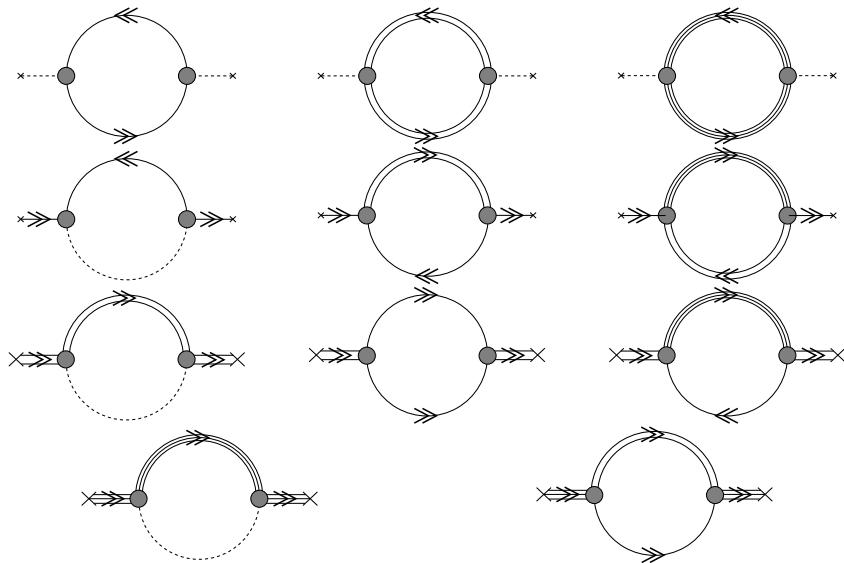


Figure: Selfenergies for Greens functions of Q-M-D-B system in 2-loop approx.

Mott Dissociation of Pions in Quark Matter

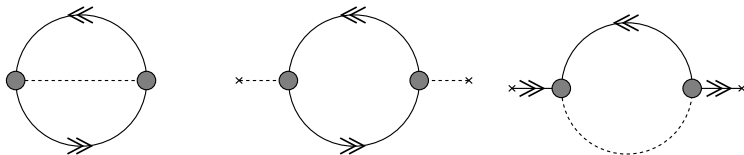


Figure: The Φ functional (left panel) for the case of mesons in quark matter, where the bosonic meson propagator is defined by the dashed line and the fermionic quark propagators are shown by the solid lines with arrows. The corresponding meson and quark selfenergies are shown in the middle and right panels, respectively.

Mott Dissociation of Pions in Quark Matter

The meson polarization loop $\Pi_M(q, z)$ enters the definition of the meson T matrix

$$T_M^{-1}(q, \omega + i\eta) = G_S^{-1} - \Pi_M(q, \omega + i\eta) = |T_M(q, \omega)|^{-1} e^{-i\delta_M(q, \omega)} ,$$

which in the polar representation introduces a phase shift $\delta_M(q, \omega) = \arctan(\Im T_M / \Re T_M)$, that results in a generalized Beth-Uhlenbeck equation of state for the thermodynamics of the consistently coupled quark-meson system

$$\Omega = \Omega_{\text{MF}} + \Omega_M ,$$

where the selfconsistent quark meanfield contribution is

$$\Omega_{\text{MF}} = \frac{\sigma_{\text{MF}}^2}{4G_S} - 2N_c N_f \int \frac{d^3 p}{(2\pi)^3} \left[E_p + T \ln \left(1 + e^{-(E_p - \Sigma_+ - \mu)/T} \right) + T \ln \left(1 + e^{-(E_p + \Sigma_- + \mu)/T} \right) \right] ,$$

The mesonic contribution to the thermodynamics is

$$\Omega_M = d_M \int \frac{d^3 k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \left\{ \omega + 2T \ln \left[1 - e^{-\omega/T} \right] \right\} 2 \sin^2 \delta_M(k, \omega) \frac{\delta_M(k, \omega)}{d\omega} ,$$

Mott Dissociation of Pions in Quark Matter

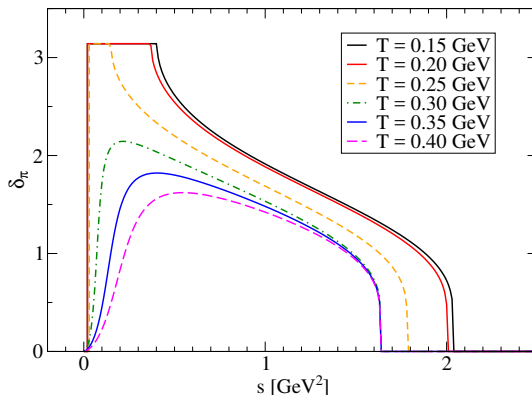


Figure: Phase shift of the pion as a quark-antiquark state for different temperatures, below and above the Mott dissociation temperature. From D.B. et al., Ann. Phys. 348 (2014) 228. See also [Poster by I. Soudi & D.B.](#)

Precursor to Mott Dissociation: Quark Pauli Blocking

$$\Omega = \text{Tr} \left\{ \ln S_q^{-1} - \Sigma_q S_q - \frac{1}{2} \ln D_\pi^{-1} + \frac{1}{2} D_\pi \Pi_\pi \right\} + \Phi[S_q, D_\pi], \quad \Phi[S_q, D_\pi] =$$



Dyson-Schwinger equations $S_q^{-1} = S_{q,MF}^{-1} - \Sigma_q$ and $D_\pi^{-1} = G_\pi^{-1} - \Pi_\pi$, resp.

$$\Sigma_q = \frac{\delta \Phi}{\delta S_q} =$$

$$\approx$$

$$+ \dots ;$$

$$\Pi_\pi = \frac{\delta \Phi}{\delta D_\pi} =$$

$$\approx$$

$$+ 2$$

$$+ \dots$$

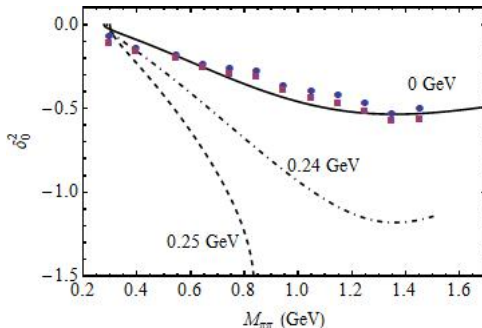
Perturbation around selfconsistent meanfield reveals quark exchange contribution to $\pi\pi$ scattering in isospin=2 channel

$$\equiv$$

$$,$$

Quark Pauli Blocking in the Pion Gas

Nonrelativistic potential model evaluation of quark exchange diagrams
 $\Rightarrow \delta_0^2$ phase shift for $\pi^+\pi^+$ scattering [Barnes & Swanson, PRD (1992)]

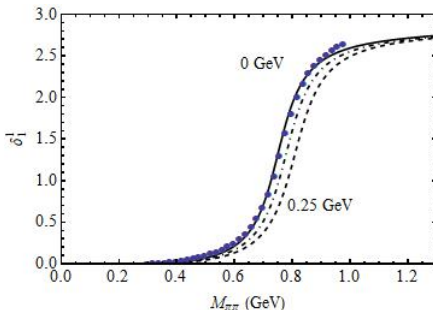
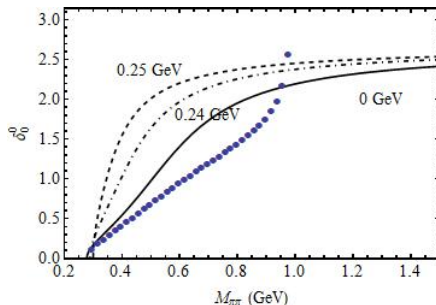


$$\sin \delta_0^2 = -\frac{\alpha_s}{9\lambda m_q^2} \sqrt{\frac{s}{\xi}} \left(1 - e^{-\lambda\xi/2} + \frac{4\lambda\xi}{3\sqrt{3}} e^{-\lambda\xi/3} \right), \quad \xi = s - 4M_\pi^2$$

Let m_q depend on temperature $\rightarrow$ chiral enhancement!

Quark Pauli Blocking: T-Dependence of Phase Shifts

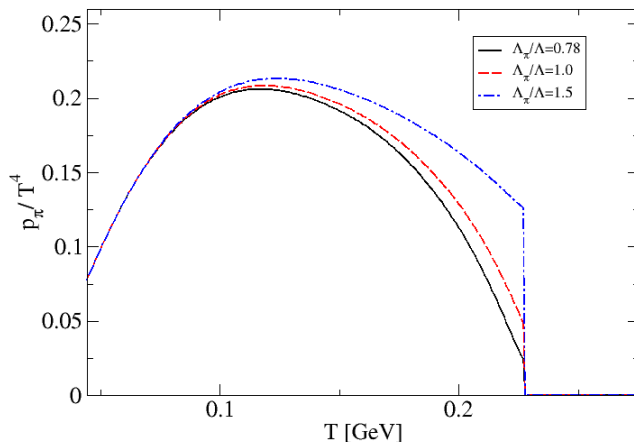
Simple Breit-Wigner parametrization of $\pi - \pi$ phase shifts in the sigma-meson (δ_0^0) and rho-meson δ_1^1 channels



Let meson masses and widths depend on temperature $\rightarrow$ chiral enhancement!

Virial expansion of the Pion Gas Pressure

Second virial coefficient in Beth-Uhlenbeck formula with $\pi - \pi$ phase shifts



Relativistic Density Functional Approach to Nuclear Matter

In case of color confinement all closed loop diagrams with Q- and D-lines vanish. The system reduces to the M-B- system. The Φ -functional becomes a density functional.

$$\Phi = \text{diagram} \Rightarrow \mathcal{U} = \text{diagram}$$

The diagram on the left is a circle with two vertices (grey dots) connected by a horizontal dashed line. The top and bottom arcs of the circle have arrows pointing clockwise. The diagram on the right is a square with a grey shaded interior labeled G_{i+j} . The top and bottom edges of the square have curly braces labeled i and j respectively, with arrows pointing outwards.

$$\Omega = T \sum_{i=n,p,\Lambda,\dots} c_i \left[\text{Tr} \ln S_{i,qu}^{-1} + \sum_{j=S,V} n_{i,j} \Sigma_{i,j} \right] + U \left[\{n_{i,S}, n_{i,V}\} \right],$$

$$\frac{\partial \Omega}{\partial n_{i,S}} = \frac{\partial \Omega}{\partial n_{i,V}} = 0, \quad i = n, p, \Lambda, \dots,$$

$$\frac{\partial U}{\partial n_{i,S}} = \Sigma_{i,S}, \quad \frac{\partial U}{\partial n_{i,V}} = \Sigma_{i,V}.$$

The baryon quasiparticle propagators fulfill the Dyson equations $S_{i,qu}^{-1} = S_{i,0}^{-1} - \Sigma_{i,S} - \Sigma_{i,V}$,

$$\text{diagram}_1 = \text{diagram}_2 + \text{diagram}_3$$

The first diagram is a horizontal line with a curly brace labeled i and "qu" below it. The second diagram is a horizontal line with a curly brace labeled i and "id" below it. The third diagram is a horizontal line with a curly brace labeled i and "id" below it on the left, a curly brace labeled i and "qu" below it on the right, and a square with a grey shaded interior labeled G_{i+j} in the middle. The top and bottom edges of the square have curly braces labeled j and "qu" above them, with arrows pointing outwards.

Quark Pauli Blocking in Hadronic Matter

Perturbative expansion around the quasiparticle Q- and D- propagators

$$\begin{array}{c} \text{---}\gg\text{---} \\ \text{---}\gg\text{---} \end{array} = \begin{array}{c} \text{---}\xrightarrow{\text{qu}}\text{---} \\ \text{---}\xrightarrow{\text{qu}}\text{---} \end{array} + \begin{array}{c} \text{qu} \\ \text{qu} \end{array} + \mathcal{O}(\Sigma^{(2)})$$

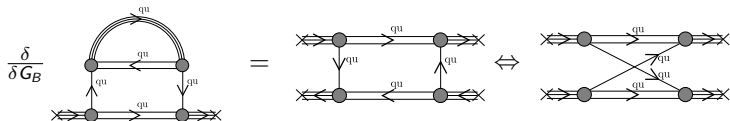
$$\begin{array}{c} \text{---}\gg\text{---} \\ \text{---}\gg\text{---} \end{array} = \begin{array}{c} \text{---}\xrightarrow{\text{qu}}\text{---} \\ \text{---}\xrightarrow{\text{qu}}\text{---} \end{array} + \begin{array}{c} \text{qu} \\ \text{qu} \end{array} + \mathcal{O}(\Sigma^{(2)})$$

Insertion into the Q-D loop diagram defining the baryon

$$\begin{array}{c} \text{---}\gg\text{---} \\ \text{---}\gg\text{---} \end{array} = \Omega(\Sigma^{(0)}) + \begin{array}{c} \text{qu} \\ \text{qu} \end{array} + \begin{array}{c} \text{qu} \\ \text{qu} \end{array} + \mathcal{O}(\Sigma^{(2)})$$

Quark Pauli Blocking in Hadronic Matter

The "new" baryon selfenergy diagrams contain one closed baryon loop, are proportional to baryon density. Functional derivative w.r.t. the baryon propagator yields effective interaction



For the diagrammatic expansion, see also K. Maeda, Ann Phys. **326** (2011) 1032. Quark Pauli blocking has been evaluated, e.g. in nonrelativistic quark models, with constant (constituent) quark mass [G. Röpke et al., PRD **34** (1986) 3499]. Here, effects of chiral symmetry restoration in a hadronic medium are taken into account. They lead to a strong enhancement of the Pauli blocking energy shift and drive the system into dissociation/deconfinement!

Note: Pauli blocking effect in a pion gas completely analogous!



Jean-Paul Blaizot

Kenji Maeda

D.B.

Helmholtz International Summer School
“Dense QCD Phases in Heavy-Ion Collisions”
Dubna, August 21 – September 4, 2010

Example: Quark Pauli Blocking in Nuclear Matter

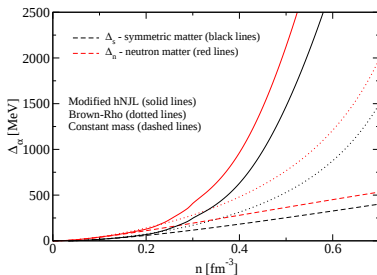
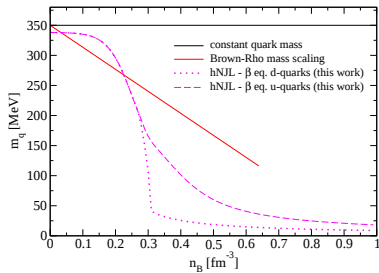


Figure: Left panel: Different quark mass dependences on the density; Right panel: Resulting Pauli blocking energy shift in symmetric matter (black lines) and in pure neutron matter (red lines).

Example: Quark Pauli Blocking in Nuclear Matter

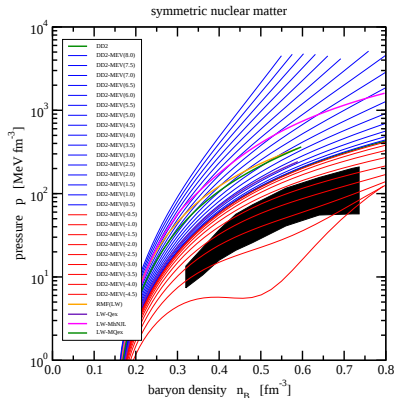


Figure: Pressure vs. density for chirally enhanced quark Pauli blocking within a linear Walecka model scheme. Different line colors stand for the quark mass scalings. For comparison the DD2 RMF model with modified excluded volume [S. Typel, EPJA 52 (2016)] is shown by blue lines (positive v -parameter) and red lines (negative v -parameter).

Example: Quark Pauli Blocking in Nuclear Matter

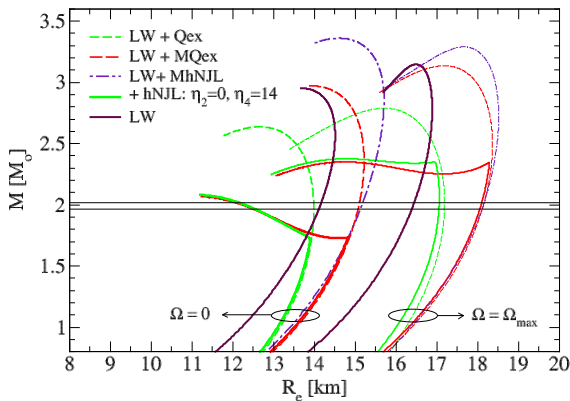


Figure: Mass vs. radius for hybrid stars resulting from a hadronic EoS with quark Pauli blocking and a higher order NJL model for quark matter. D. Blaschke, H. Grigorian, G. Röpke, in preparation for MDPI Particles (2018).

Relativistic Density Functional Approach to Quark Matter

$$\mathcal{Z} = \int \mathcal{D}\bar{q}\mathcal{D}q \exp \left\{ \int_0^\beta d\tau \int_V d^3x [\mathcal{L}_{\text{eff}} + \bar{q}\gamma_0 \hat{\mu}q] \right\} ,$$

$$U(\bar{q}q, \bar{q}\gamma_0 q) = U(n_s, n_v) + (\bar{q}q - n_s)\Sigma_s + (\bar{q}\gamma_0 q - n_v)\Sigma_v + \dots ,$$

$$\Omega = -T \ln \mathcal{Z} = \Omega^{\text{quasi}} + U(n_s, n_v) - n_s \Sigma_s - n_v \Sigma_v .$$

The quasi-particle term (for the case of isospin symmetry and degenerate flavors)

$$\Omega^{\text{quasi}} = -2N_c N_f T \int \frac{d^3p}{(2\pi)^3} \left\{ \ln \left[1 + e^{-\beta(E^* - \mu^*)} \right] + \ln \left[1 + e^{-\beta(E^* + \mu^*)} \right] \right\}$$

can be calculated by using the ideal Fermi gas distribution for quarks with the quasiparticle energy $E^* = \sqrt{p^2 + M^2}$, the effective mass $M = m + \Sigma_s$ and effective chemical potential $\mu^* = \mu - \Sigma_v$. The self energies are determined by the density derivations

$$\Sigma_s = \frac{\partial U(n_s, n_v)}{\partial n_s} , \quad \text{and}$$

$$\Sigma_v = \frac{\partial U(n_s, n_v)}{\partial n_v} .$$

In this approach the stationarity of the thermodynamical potential

$$0 = \frac{\partial \Omega}{\partial n_s} = \frac{\partial \Omega}{\partial n_v}$$

is always fulfilled.

Relativistic Density Functional Approach to Quark Matter

To capture the phenomenology of a confining meanfield (string-flip model), the following density functional of the interaction is adopted,

$$U(n_s, n_v) = D(n_v)n_s^{2/3} + an_v^2 + \frac{bn_v^4}{1 + cn_v^2}.$$

The first term captures aspects of (quark) confinement through the density dependent scalar self-energy,

$$\Sigma_s = \frac{2}{3}D(n_v)n_s^{-1/3},$$

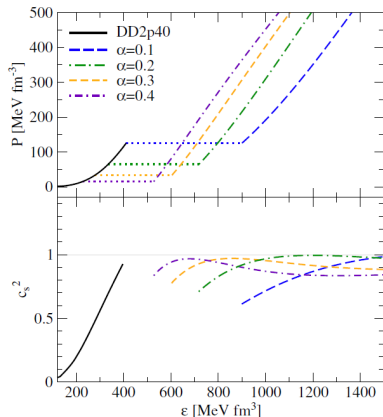
defining the effective quark mass $M = m + \Sigma_s$. We also employ higher-order quark interactionsto obey the observational constraint of $2 M_\odot$. The denominator in the last term of Equation (26) guarantees that the speed of sound $c_s = \sqrt{\partial P / \partial \varepsilon}$ does not exceed the speed of light). All terms in Equation (26) that contain the vector density contribute to the shift defining the effective chemical potentials $\mu^* = \mu - \Sigma_v$, where

$$\Sigma_v = 2an_v + \frac{4bn_v^3}{1 + cn_v^2} - \frac{2bcn_v^5}{(1 + cn_v^2)^2} + \frac{\partial D(n_v)}{\partial n_v}n_s^{2/3}.$$

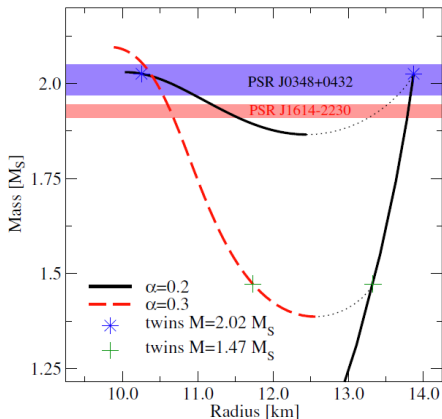
The reduction of the string tension $D(n_v) = D_0\phi(n_v; \alpha)$ is modeled via a Gaussian function of the baryon density n_v ,

$$\phi(n_v; \alpha) = \exp[-\alpha(n_v \cdot \text{fm}^3)^2],$$

Hybrid EoS: Third Family of Compact Stars & Mass Twins

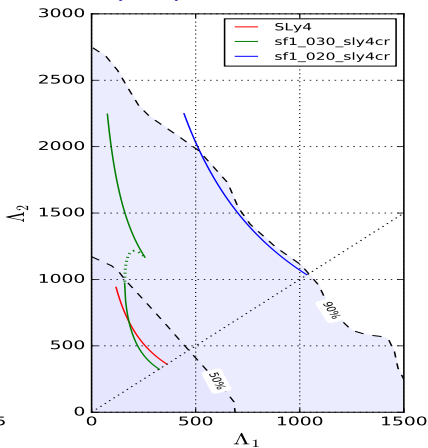
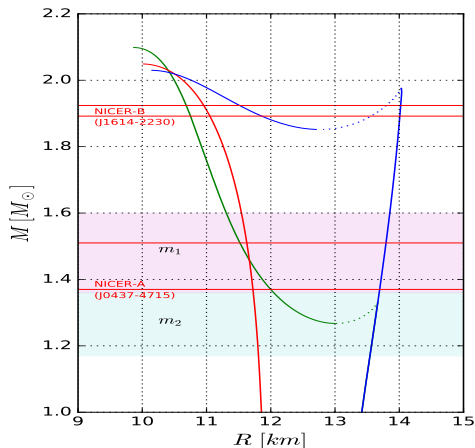


KALTENBORN, BASTIAN, and BLASCHKE



PHYSICAL REVIEW D **96**, 056024 (2017)

Was GW170817 NOT a neutron star (NS) - NS merger ?



GW170817 can be explained as a hybrid star (HS) - HS merger for a low-mass twin EoS as well as a NS - NS merger for a soft nuclear matter EoS. If NICER measures $R_{J0437} \geq 14$ km $\Rightarrow$ evidence for a strong phase transition !

Summary

- cluster virial expansion developed for sunset-type Φ functionals made of cluster Green's functions and a cluster T-matrix
- cluster Φ functional approach to quark-meson-diquark-baryon system developed and example for meson dissociation outlined
- quark Pauli blocking in hadronic matter is contained in the approach
- selfconsistent density-functional approach to quark matter with confinement and chiral symmetry breaking obtained as limiting case
- applications to nuclear clustering and quark deconfinement in the astrophysics of supernovae and compact stars as well as in heavy-ion collisions

Outlook

- cluster virial expansion for quark-hadron matter as a relativistic density functional with bound state formation and dissociation
- Ginzburg-Landau-type density functional for the QCD phase diagram besides the one for the liquid-gas phase transition in nuclear matter.