Conundrums at Finite Density

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Introduction

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- Investigating these models in extreme environments led to a variety of phase diagrams of strongly interacting matter.

- QCD formulated on a space-time lattice has yielded a more firm guidance in altering these pictures. Nevertheless, there are conundrums at finite density, many unrelated to the latticization, which still pose significant hurdles.

The Early Picture

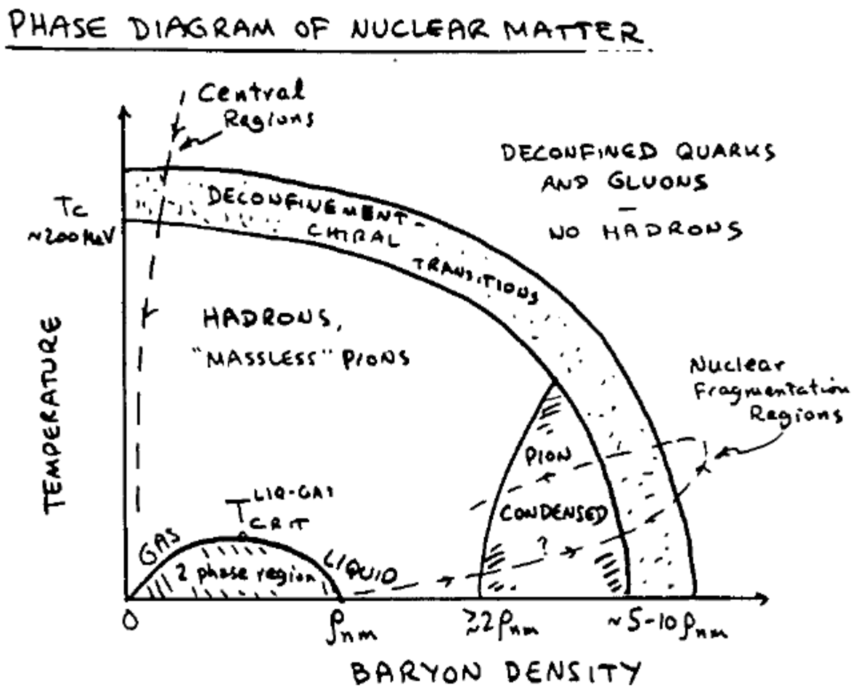

Figure 1. Phase diagram of nuclear matter in equilibrium, and how it can be explored in ultrarelativistic heavy ion collisions, from the 1983 NSAC Long Range Plan [22].

G. Baym, Quark Matter 2001, *arXiv:hep-ph/0104138v2.*

QCD Phase diagram

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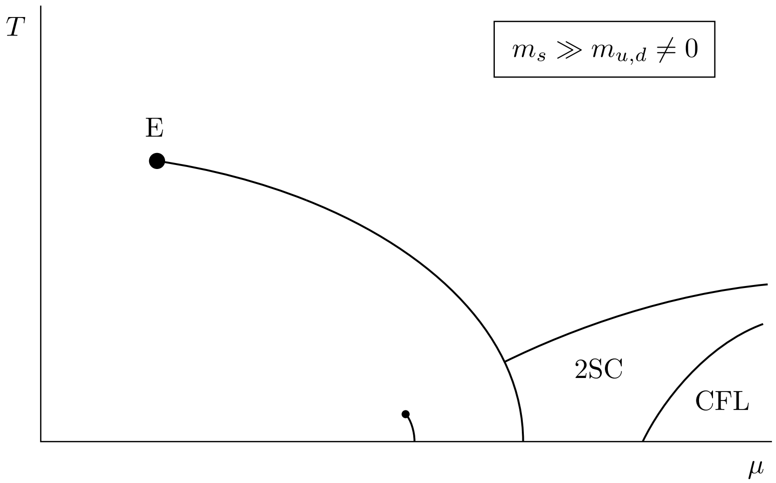

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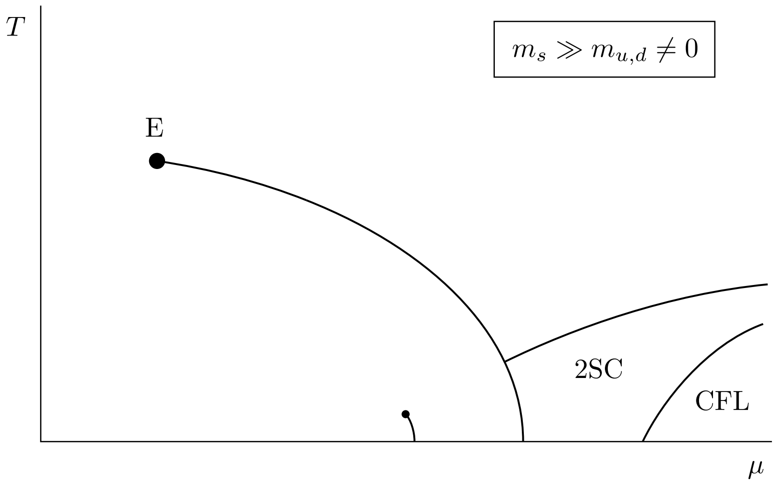

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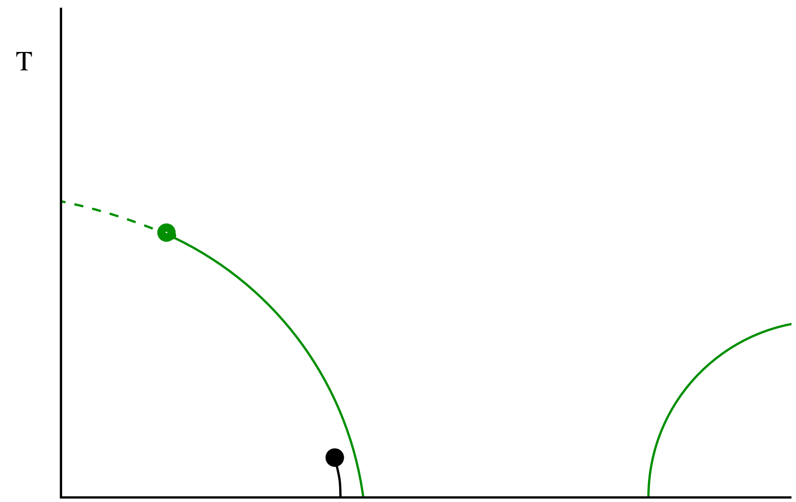

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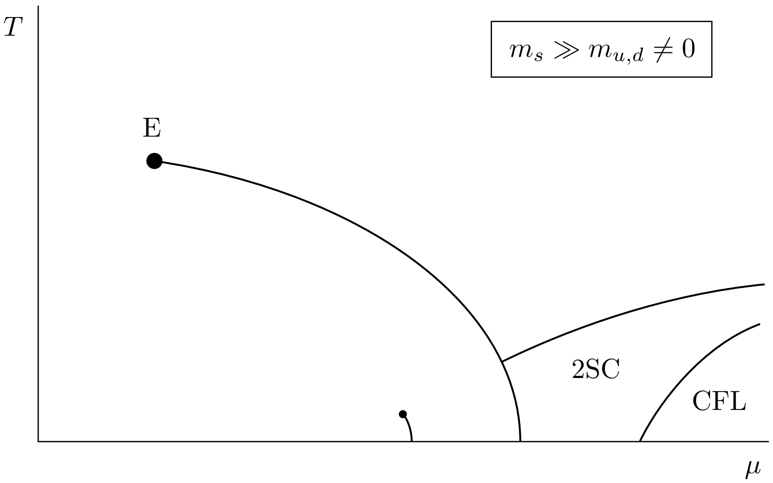

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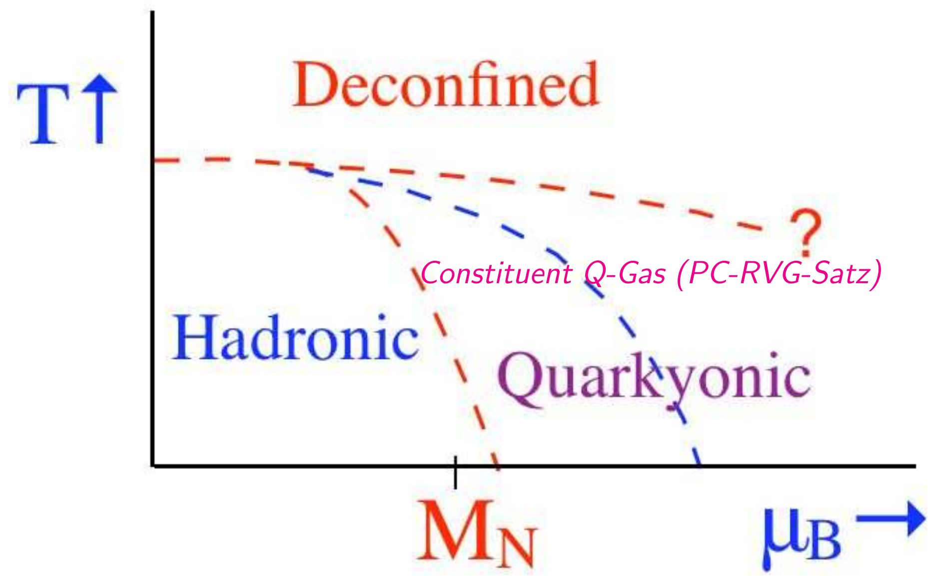

The $\mu \neq 0$ problems : I. Divergences

♠ Chemical potential introduced on the lattice by multiplying the forward [backward] time-like links with $f(\mu a)$ [$g(\mu a)$].

♢ While $f(\mu a) = 1 + \mu a$ and $g(\mu a) = 1 - \mu a$ defines the naıve/linear fermionic action for finite density, $f(\mu a) = \exp(\mu a)$ is the popular exponential choice *(Hasenfratz-Karsch, PLB 1983 & Kogut et al., PRD 1983)* along with $f(\mu a) = (1 + \mu a)/\sqrt{(}1 - \mu^2 a^2)$ as another choice *(Bilić-Gavai, EPJC 1984)*.

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♣ Easy to show that one has the following form of the energy density and quark number density in the free case.

$$
\begin{aligned}
\epsilon &= c_0 a^{-4} + c_1 \mu^2 a^{-2} + c_3 \mu^4 + c_4 \mu^2 T^2 + c_5 T^4 \\
n &= d_0 a^{-3} + d_1 \mu a^{-2} + d_3 \mu^3 + d_4 \mu T^2 + d_5 T^3 .
\end{aligned} \tag{1}
$$

♠ Subtracting off vacuum contribution at $T = 0 = \mu$, eliminates the leading divergence in each case in the $a \to 0$ limit.

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♡ Both the other two actions have been chosen to have $c_1 = 0 = d_1$, and thus lead to finite results in the continuum limit. True as long as $f \cdot g = 1$ *(Gavai, PRD 1985)*.

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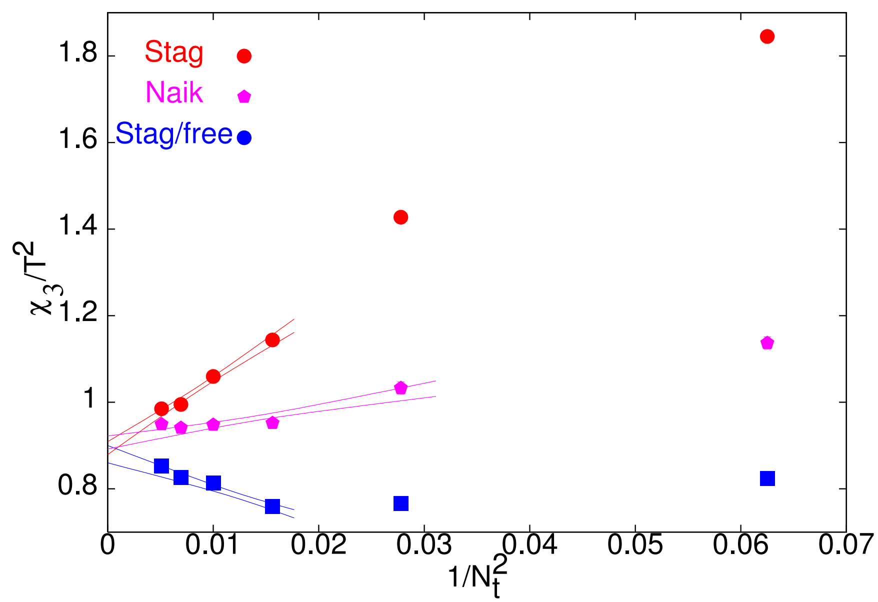
♡ Numerical computations showed that this holds for the non-perturbative interacting case as well *(Gavai-Gupta PRD 67, 034501 (2003))* :



♢ Why are there three lattice actions when QCD has one ?

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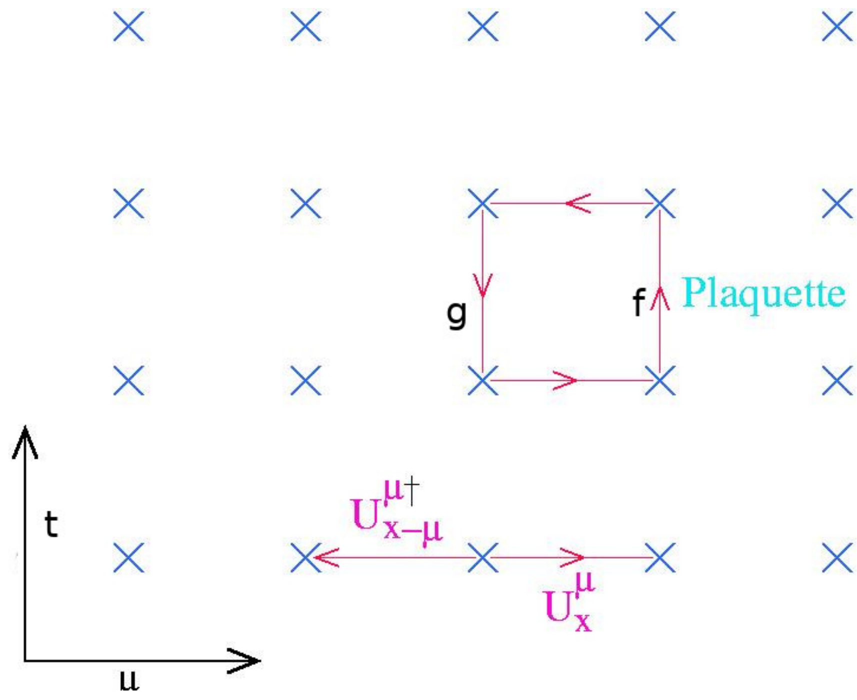
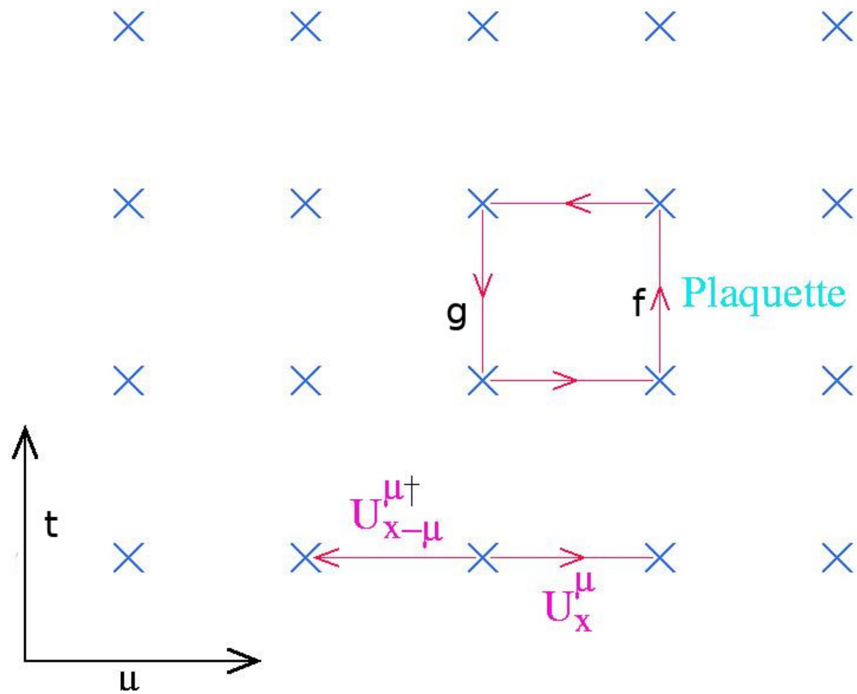
♣ Problem : Only for the linear μ-case one has a conserved charge on the lattice.

♠ With the other functions above, one has **no** conserved charge on the lattice anymore ! Alternatively $Z \neq \exp(-\beta[\hat{H} - \mu\hat{N}])$ on the lattice for them. Possible only in the continuum limit of $a \to 0$.

♣ One cannot define an *exact* canonical partition function on lattice from the Z defined this way. $Z = \sum_n z^n Z_n^C$ on the lattice only for the naıve action.



$\diamondsuit$ Any μ-dependence for $\mathcal{Z}$ solely due to loops with time-like links, and hence $\propto (f \cdot g)^l$, with $l = 1$ in Figure.



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♠ Quark loops of **all** sizes and types contribute for the naıve case of f,g $= 1 \pm \mu a$, as is indeed the case in the continuum.

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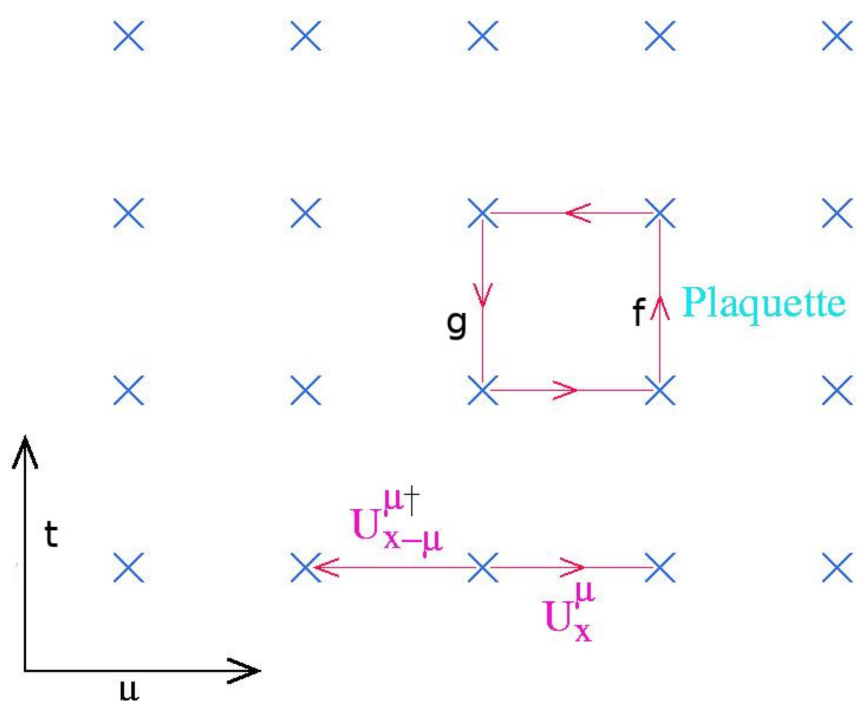

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♡ However, only quark loops winding around the T direction contribute to μ_B dependence for other cases since $f \cdot g = 1$.

♣ Moreover, as $a \to 0$ small quark loops must start contributing. How is this possible? Do the small loops sum up to zero/constant ?

♣ The same universality violation issue again ! At least, crucial to verify that it is obeyed for all three actions.

Divergences exist in Continuum too

- It turns out that contrary to the common belief, the free theory divergences are **NOT** a lattice artifact. They exist in continuum too.

- Indeed lattice regulator simply makes it easy to spot them. Using a momentum cut-off Λ in the continuum theory, one can show the presence of $\mu\Lambda^2$ terms in number density easily *(Gavai-Sharma, PLB 2015)*.

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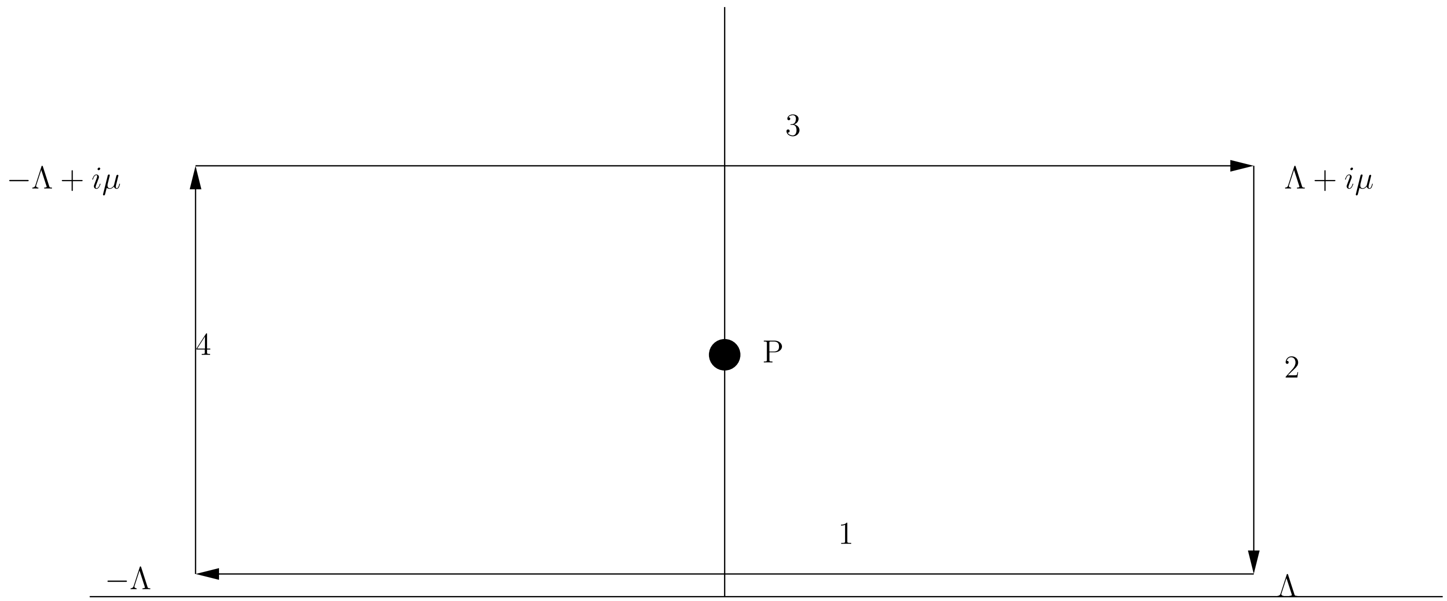
- Evaluating the quark number density, n in the momentum space for the free quark gas, one has

$$n = \frac{2iT}{V} \sum_m \int \frac{d^3p}{(2\pi)^3} \frac{(\omega_m - i\mu)}{p^2 + (\omega_m - i\mu)^2} \equiv \frac{2iT}{V} \int \frac{d^3p}{(2\pi)^3} \sum_{\omega_m} F(\omega_m, \mu, \vec{p}), \quad (2)$$

where $p^2 = p_1^2 + p_2^2 + p_3^2$ and $\omega_m = (2m+1)\pi T$.

- In the usual contour method, the sum over m gets replaced as an integral in the complex ω-plane. Together with the subtracted vacuum (μ=0) contribution, one has in the complex ω-plane:



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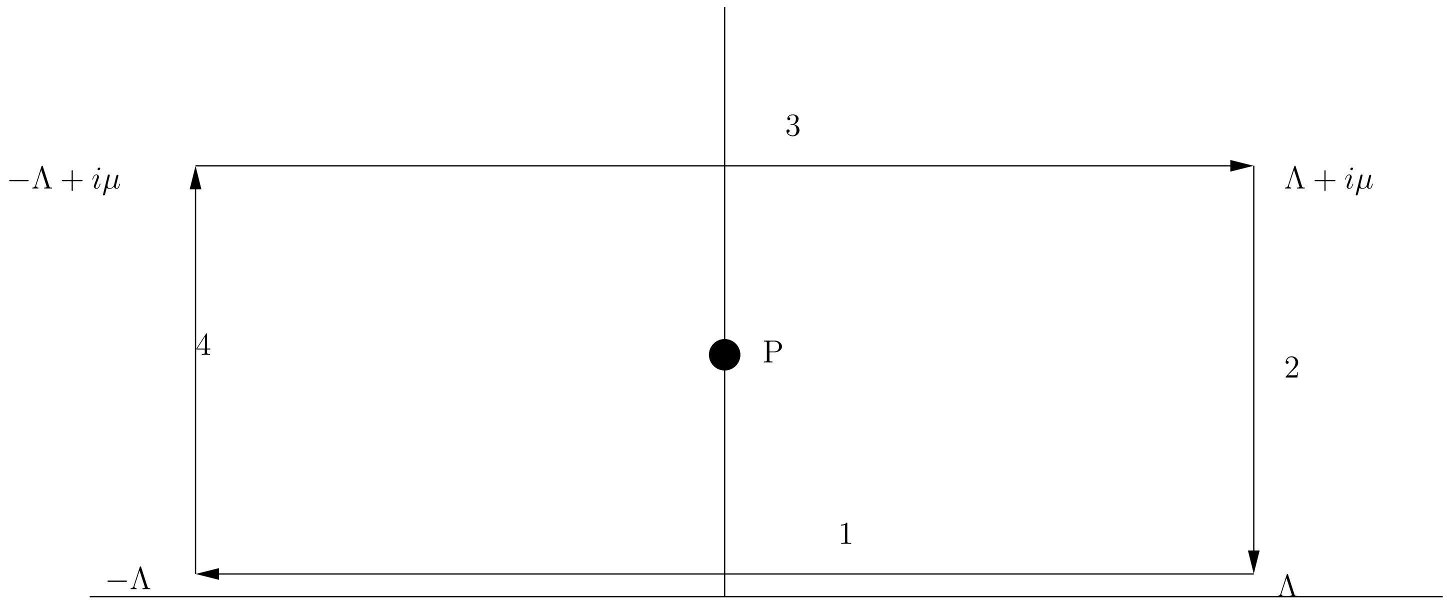

- Introduce a cut-off Λ for all 4-momenta at $T = 0$ for a careful evaluation of the divergent arms 2 & 4 contributions in figure above. The $\mu\Lambda^2$ terms arise from the arms 2 & 4. *(Gavai-Sharma, PLB 2015)* :

$$\int \frac{d^3p}{(2\pi)^3}\left(\int_2 + \int_4\right)\frac{d\omega}{\pi}\frac{\omega}{p^2+\omega^2} = -\frac{1}{2\pi}\int \frac{d^3p}{2\pi^3}\ln\left[\frac{p^2+(\Lambda+i\mu)^2}{p^2+(\Lambda-i\mu)^2}\right]. \quad (3)$$

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- Note also that the arms 2 & 4 make a finite contributions to the μ^3 term as well.

- Ignoring the contribution from the arms 2 & 4 amounts to a subtraction of the 'free theory divergence' in continuum !

- Since the divergences exist in the continuum, and are simply subtracted, one may follow the prescription of subtracting the free theory divergence by hand on the lattice as well.

Testing the idea

- In order to test whether the divergence is truly absent in simulations as well, one needs to take the continuum limit $a \to 0$ or equivalently $N_t \to \infty$ at fixed $T^{-1} = aN_t$.

- This was tested for quenched QCD at $T/T_c = 1.25$ & 2 *(Gavai-Sharma, PLB 2015)*. For $m/T_c = 0.1$, $N_t = 4$, 6, 8, 10 and 12 lattices were employed. On 50-100 independent configurations different susceptibilities were computed.

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- $1/a^2$-term for free fermions on the corresponding $N^3 \times \infty$ lattice was subtracted from the computed values of the susceptibility.

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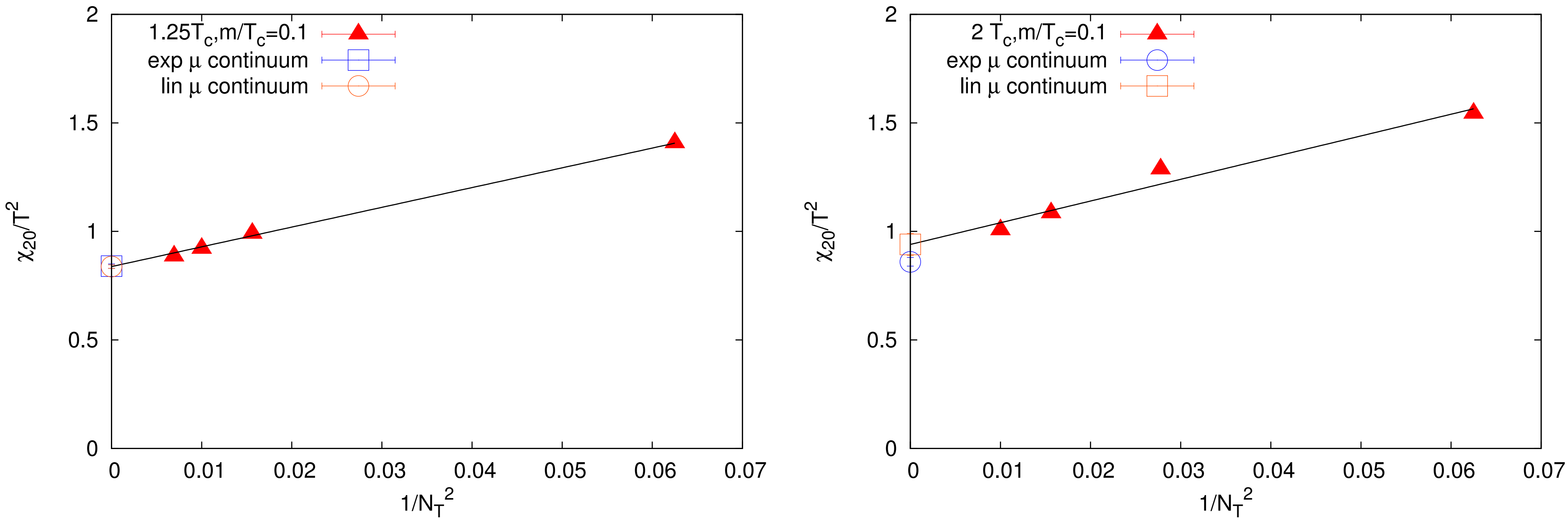
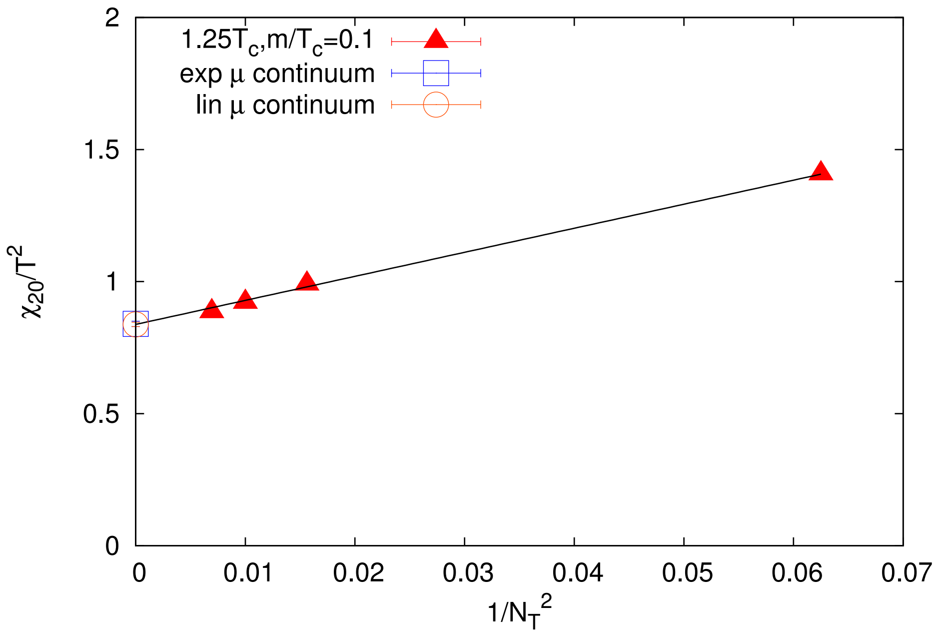
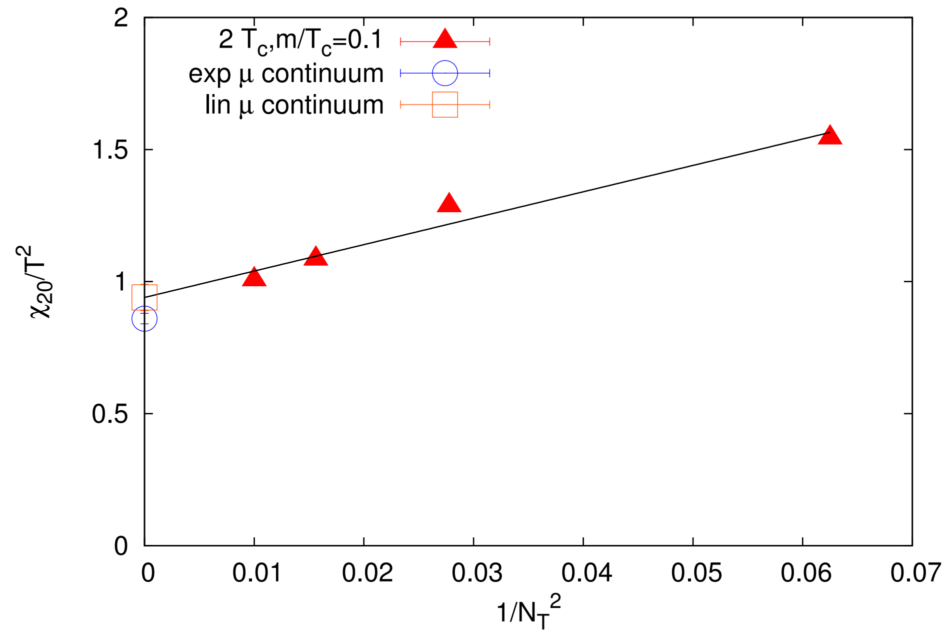


- Absence of any quadratically divergent term is evident in the positive slope of the data.





- Absence of any quadratically divergent term is evident in the positive slope of the data.

- Furthermore, the extrapolated continuum result coincides with the earlier result obtained with the $\exp(\pm a\mu)$ action *(Swagato Mukherjee PRD 2006)*.

The $\mu \neq 0$ problem : II. Quark Type

- Mostly staggered quarks used in these simulations. Broken flavour and spin symmetry on lattice. Moreover, NO flavour singlet $U_A(1)$ symmetry or anomaly. Critical point needs $N_f = 2$ and anomaly to persist by T_c *(Pisarski-Wilczek PRD 1984)*.

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- This definition is technically a bit problematic but it was shown to have no divergences in the free theory *(Gattringer & Liptak PRD 2007;Banerjee, Gavai & Sharma PRD 2008)*.

- Unfortunately the BW-prescription breaks the lattice chiral symmetry, leaving us without any order parameter at any finite density. *(Banerjee, Gavai & Sharma PRD 2008; PoS (Lattice 2008); PRD 2009)* Furthermore, it has μ-dependent anomaly unlike in continuum QCD *(Gavai & Sharma PRD 2010)*.

- Luckily, an action with continuum-like (flavour & spin) symmetries for quarks at nonzero μ and T has been proposed already. *(Gavai & Sharma , PLB 2012)*.

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- Key Idea : The massless continuum QCD action for nonzero μ can be written explicitly as sum over right and left chiral modes of quarks, thus exhibiting manifest chiral symmetry at nonzero μ.

- Such chiral projections can be defined for the Overlap quarks. Use them to construct the action at nonzero μ. It does have the exact chiral invariance on the lattice ! Then order parameter exists for the entire T-μ phase diagram. *(Gavai & Sharma , PLB 2012)*.

- Using Domain Wall formalism, it was also shown why this is physically the right thing to do: It counts only the physical (wall) modes as the baryon number while the BW action includes all the unphysical heavy modes as well.

- Note, however, this chirally invariant Overlap action with nonzero μ is in the linear form, i.e., with divergences. Interestingly, even the exponential form leads to divergences in this case *[Narayanan-Sharma JHEP 1110(2011)151]*.

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- Recently, it has been shown that SLAC fermions at finite density also need a linear form, and it too possesses these divergences *(Lenz et al. arXiv:2007.08382)*.

The $\mu \neq 0$ problem : III. Complex Measure

♠ Computations can be done IF Det^{N_f} $M > 0$ for all sets of $\{U\}$. However, Det M is a complex number for all $\mu \neq 0$: The Phase/sign problem

♡ Again this is a problem inherited from continuum QCD itself.

The $\mu \neq 0$ problem : IV. Topology

♠ Instanton vacuum has been shown to lead to chiral symmetry breaking and chiral phase transition *[Diakonov, hep-ph/9602375; Schäfer-Shuryak, RMP 1998]*.

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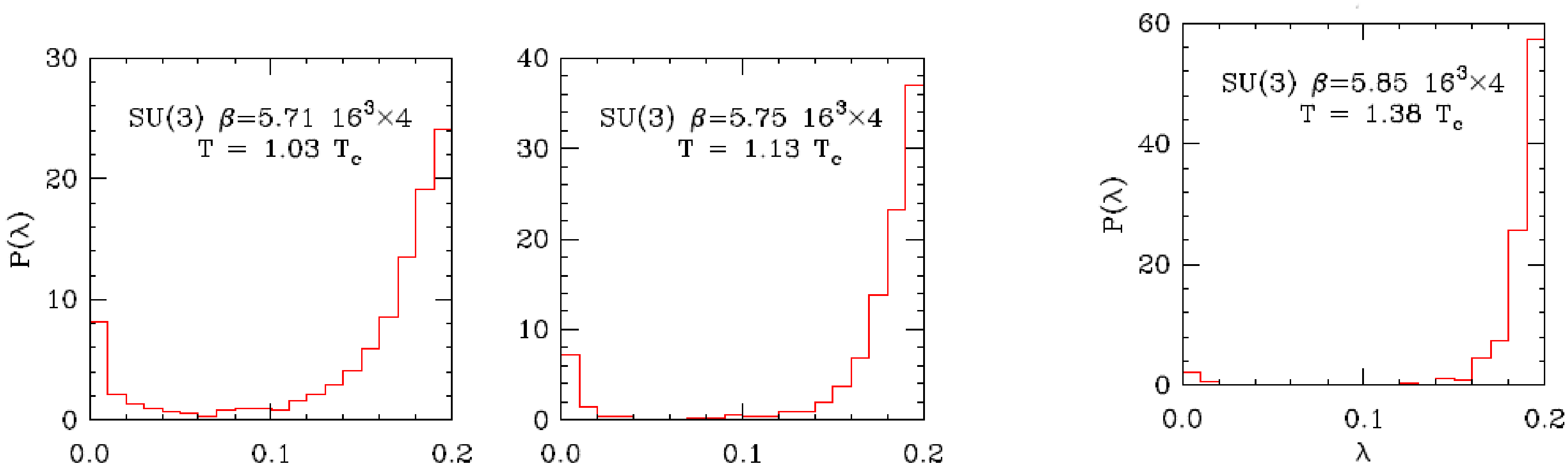
♢ Number of low eigen modes do get depleted as $T \uparrow$. *(Edwards-Heller-Kiskis-Narayanan, PRL '99, NPB (PS) '00, PRD '01; Gavai-Gupta-Lacaze, PRD '02)*



♠ What about finite μ ? This question has been addressed at nonzero isospin density in QCD as well as two colour QCD, both of which do not have a sign problem.

♡ A lot of work on both cases has been done in studying the phase structure *[Kogut-Sinclair PRD 2002; Brandt-Endrődi-Schmalzbauer PRD 2018; Kogut-Sinclair-Hands-Morrison PRD 2001; Lida-Itou-Lee JHEP 2020]*
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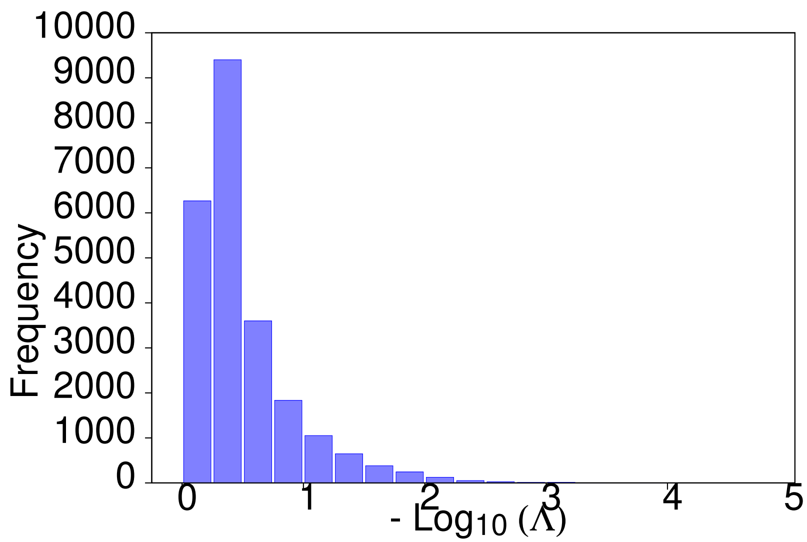
♢ Spectra of low modes, on the other hand, seems unaffected even as chiral symmetry is restored. Chiral restoration decoupled from topology?

♣ Very little/no change is seen the number of zero modes or topological susceptibility. *[Bali-Endrődi-Gavai-Mathur, 1610.00233, Lat2017; Iida-Itou-Lee JHEP 2020]*

♡ Shown are the eigenvalue distribution on the log scale to see if the near-zero modes have any difference in the two phases for nonzero isospin case

[Bali-Endrődi-Gavai-Mathur, Act. Phy. Pol. Supl. 2017]



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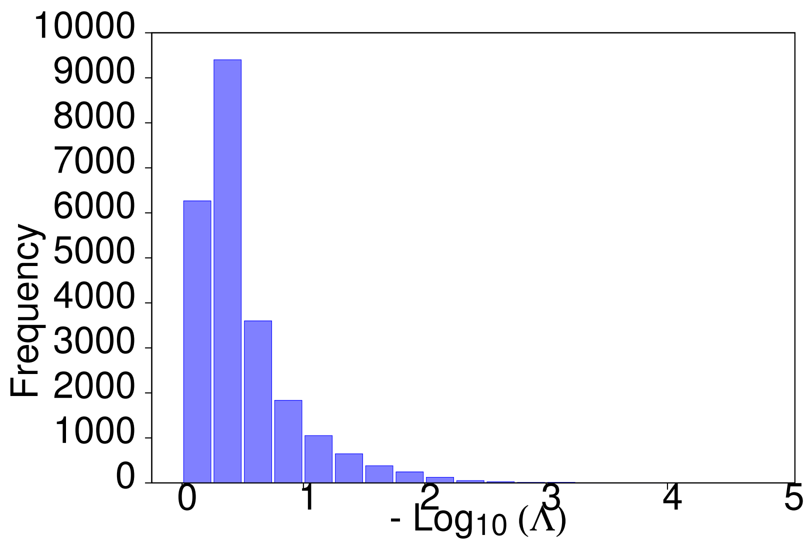

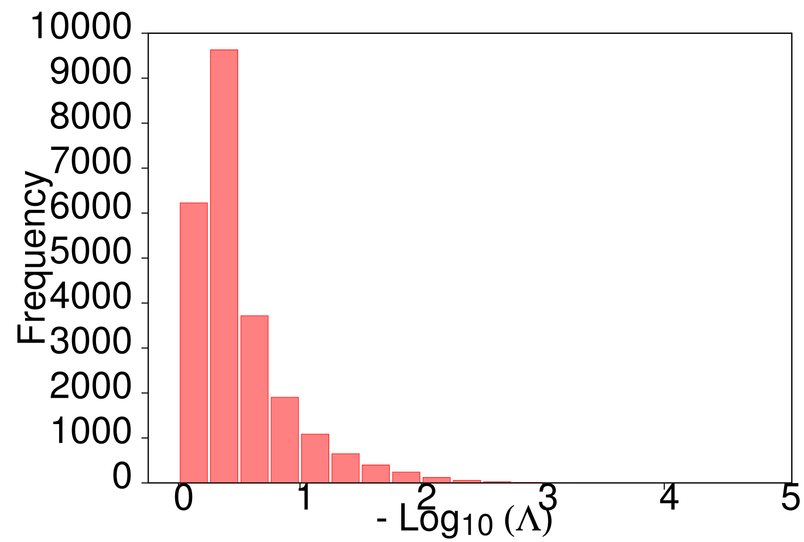

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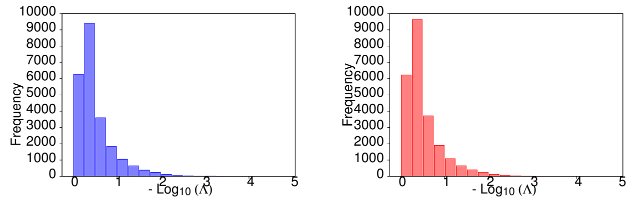
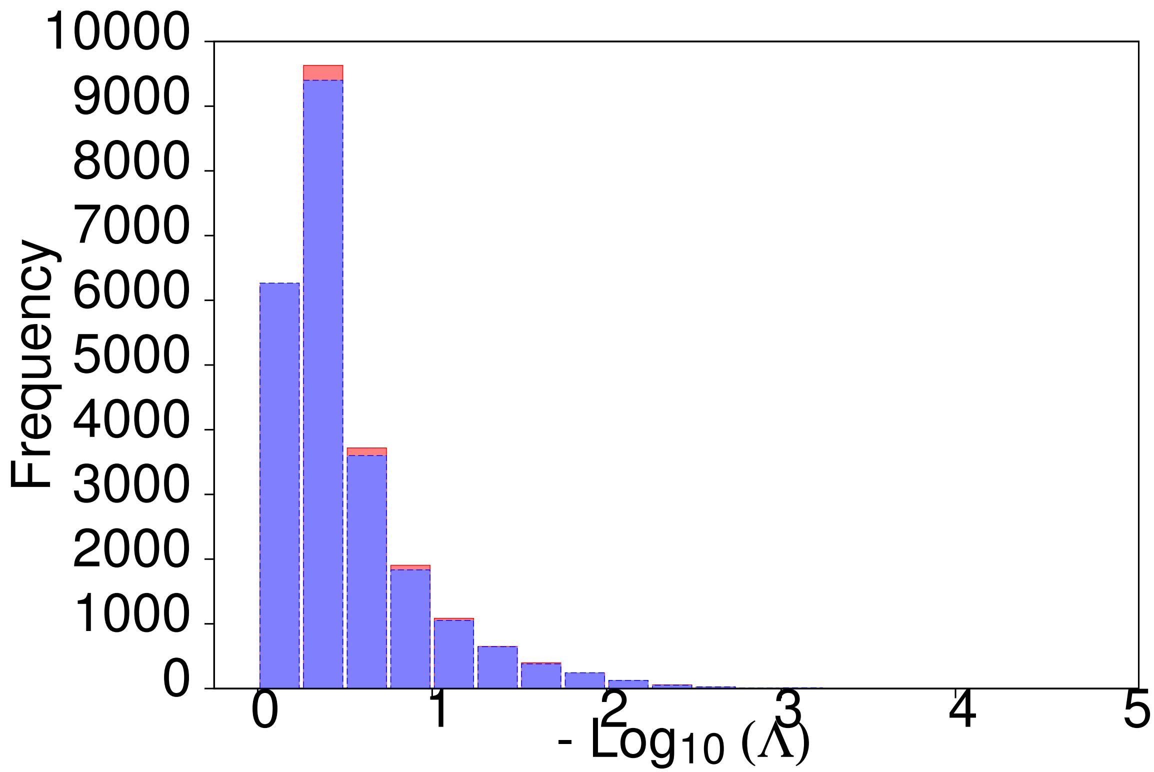
♡ $\mu_I/\mu_I^c = 1.5$: Similarity in the distribution as in the lower phase clearly indicated.





♡ No visible difference in the near-zero mode distributions.

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• For $T \neq 0$, Gavai-Gupta-Lacaze (PRD '02) found

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What about Zero Modes?

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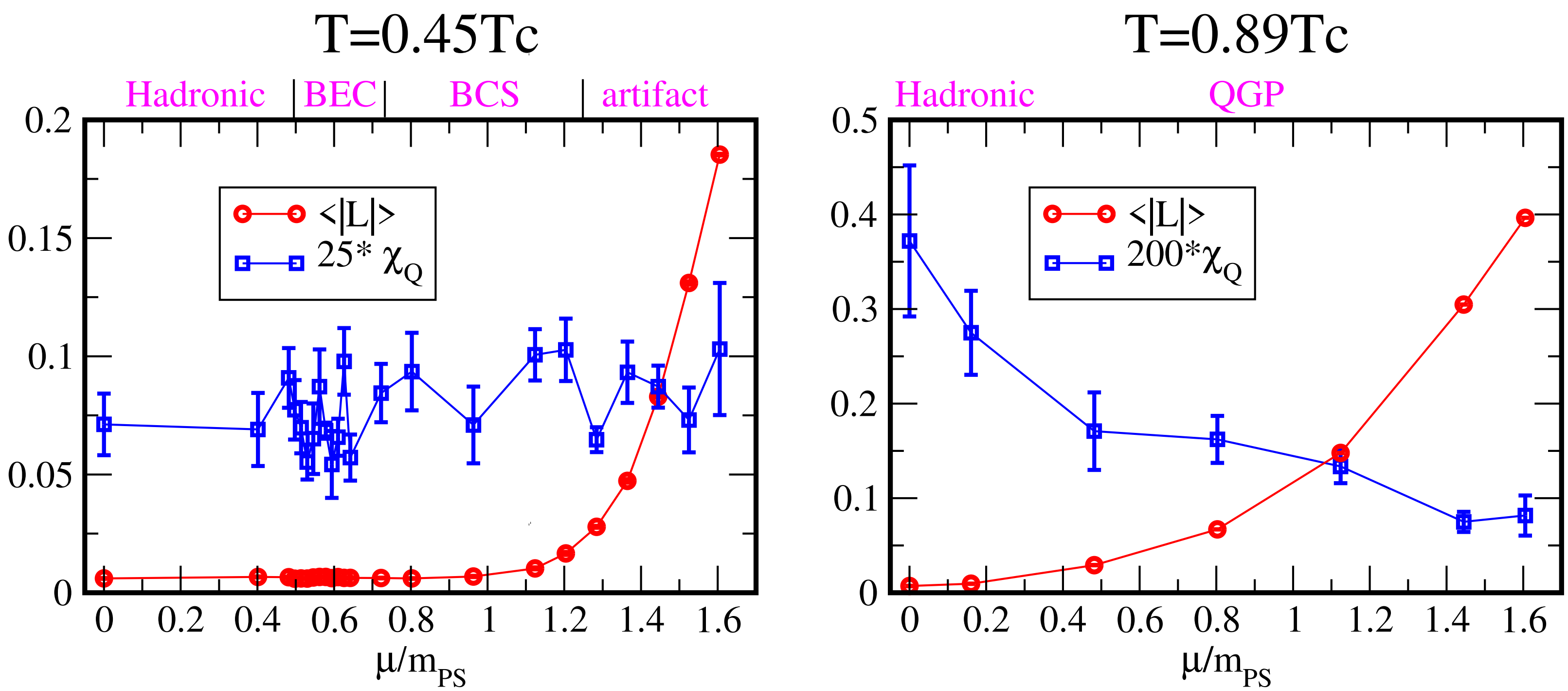
• For $\mu_I \neq 0$, one finds for *same* number of configs (50) :

μ_I/μ_I^c	$N_{zero}^{0.11}$	$N_{zero}^{0.44}$
0.5	426	477
1.5	451	332
3.0	437	396
4.0	–	562

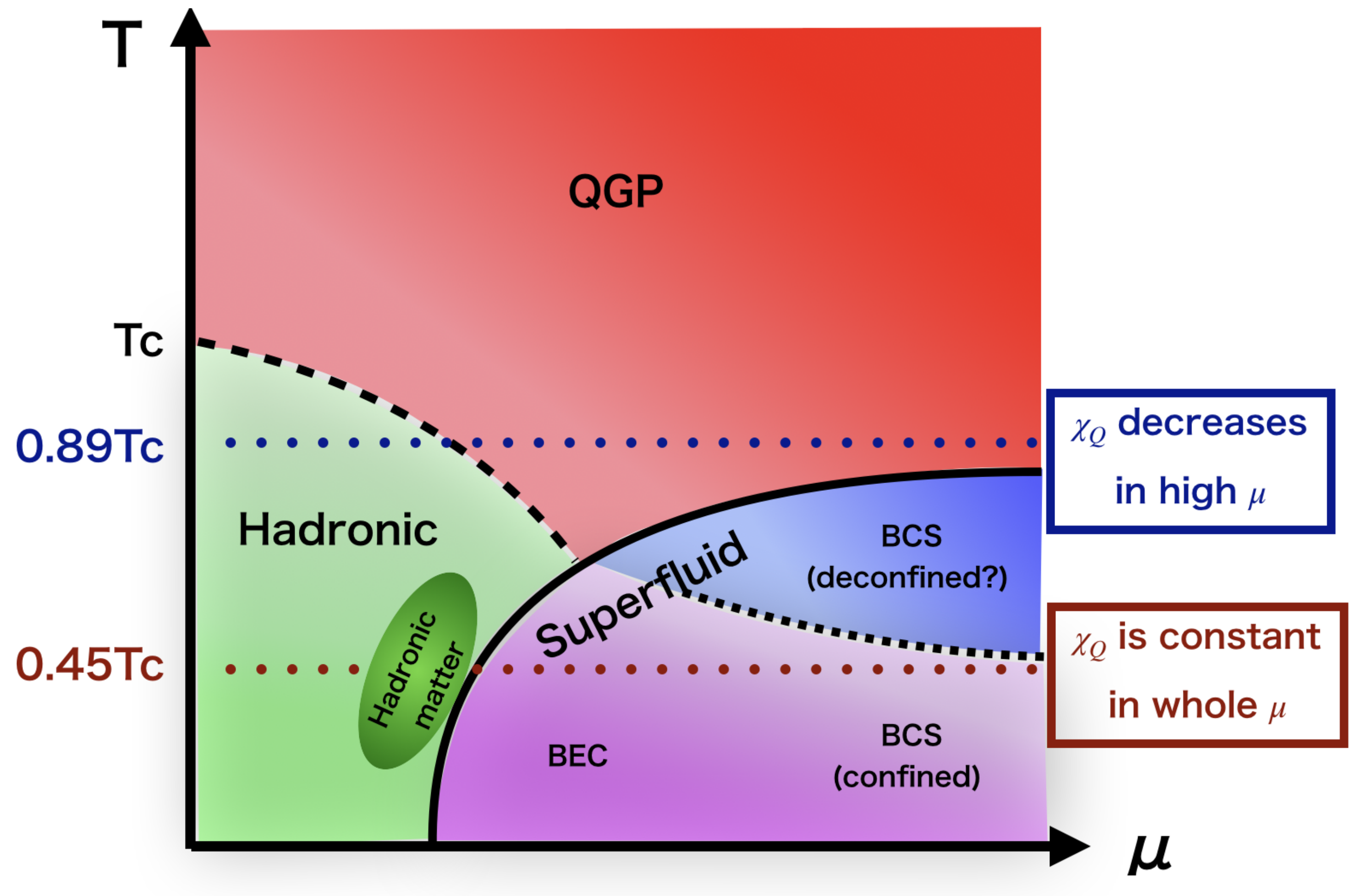
• No variation across μ_I^c for $\lambda/m_{ud} = 0.11$ & a mild dip for $\lambda/m_{ud} = 0.44$ (25% reduction at μ_I/μ_I^c=1.5)

Topology for Two-Color QCD

♡ Iida-Itou-Lee *arXiv:1910.07872* have similar results for topology for two-color QCD:



♠ Interestingly, they report both i)the decrease in χ_t in going to the high T phase, and ii) no change in the low T phase as μ is changed.



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Summary

- Most conundrums, including the μ-dependent divergence, are not due to latticization. Indeed, lattice only reproduces faithfully what exists in the continuum field theory.
- Subtraction of free theory divergences suffices even nonperturbatively.
- Chiral invariance crucial for critical point investigations but insisting on it for overlap quarks at finite density seems to always lead to a linear μ-dependent action.
- Simulations suggest that the distribution of Q in μ_I & $\mu^{N_c=2}$ across the phase transition, where the chiral condensate drops, changes very little in the low T phase in contrast to the change to high T phase, which exhibits an (exponential) fall-off. Possibly separation of χSB and confinement phase?

- Why did this remain unnoticed, even in text books ?
- Often one uses T as the cut-off in analytic computations and does frequency sums on $\omega_m = (2\pi m + 1)T$ *first* *[Kapusta-Gale Book]*. Indeed, a computation at $T = 0$ reveals it.
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- The leading divergence term from the arms 2 & 4 do cancel. One needs to regulate the momentum integrals first to spot the sub-leading ones.
- Since these divergences exist in the continuum, and are simply subtracted for the free theory, one may follow the prescription of subtracting the free theory divergence by hand on the lattice as well.