The equations of numerical relativity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$ Einstein eqs.

$$\nabla_\mu T^{\mu\nu} = 0$$ en. mom eqs.

$$\nabla_\mu (\rho u^\mu) = 0$$ cons. of rest mass.

$$p = p(\rho, \epsilon, Y_e, \ldots)$$: EoS

$T_{\mu\nu}$ is representative of the matter, field component of the spacetime

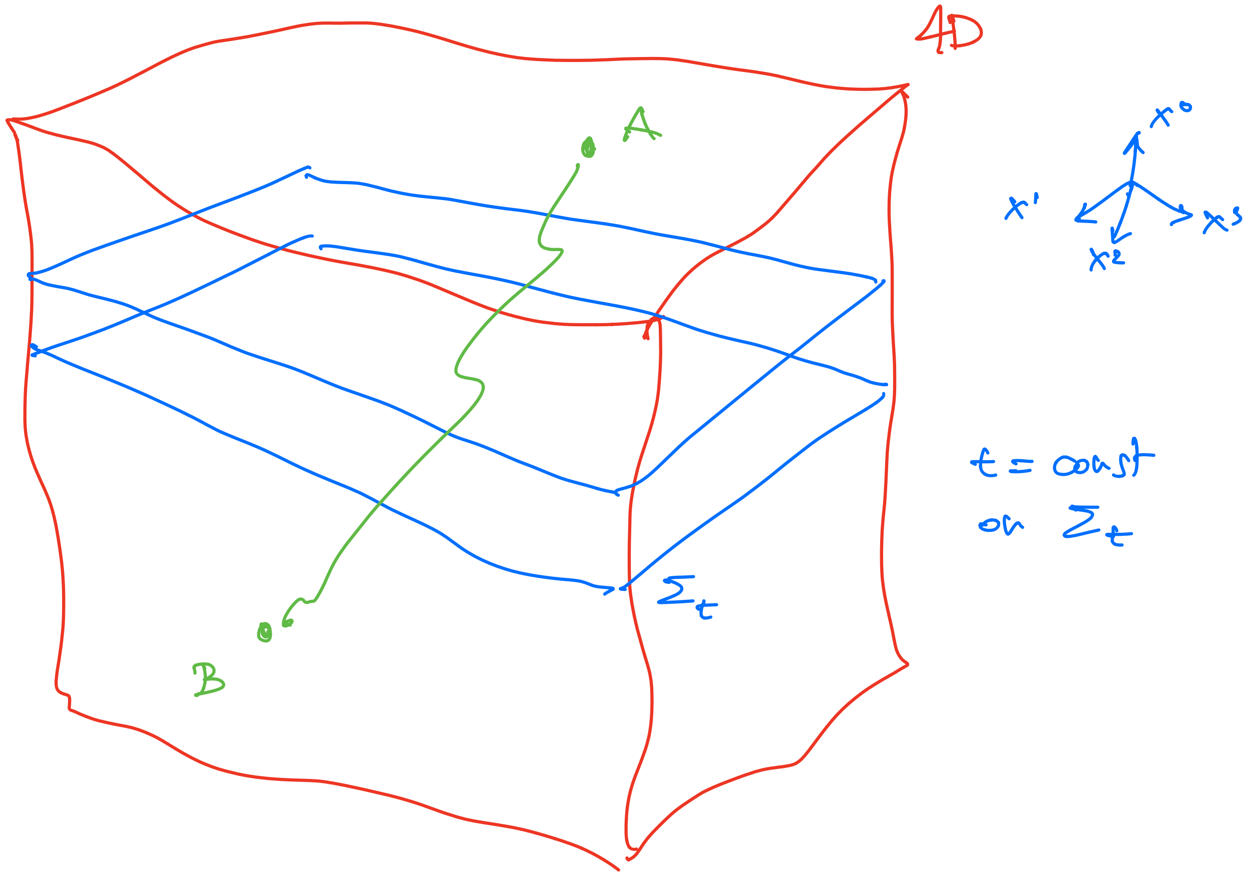
$$T^{\mu\nu} = T^{\mu\nu}_{\text{fluid}} + \underbrace{T^{\mu\nu}_{EM}}_{} + T^{\mu\nu}_{SF} + \ldots$$

new conservation laws.

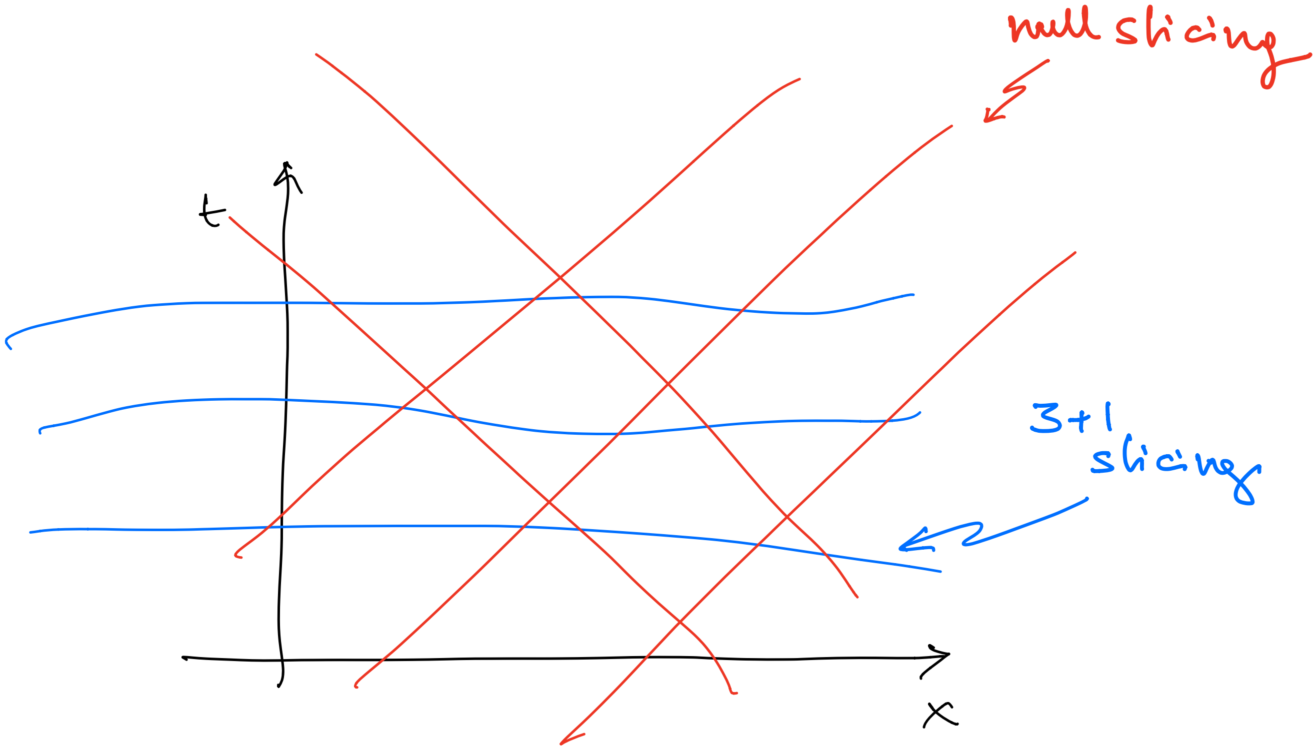
Ex: $\nabla_\nu F^{\nu\mu} = I^\mu$, $\nabla_\mu {}^*F^{\mu\nu} = 0$

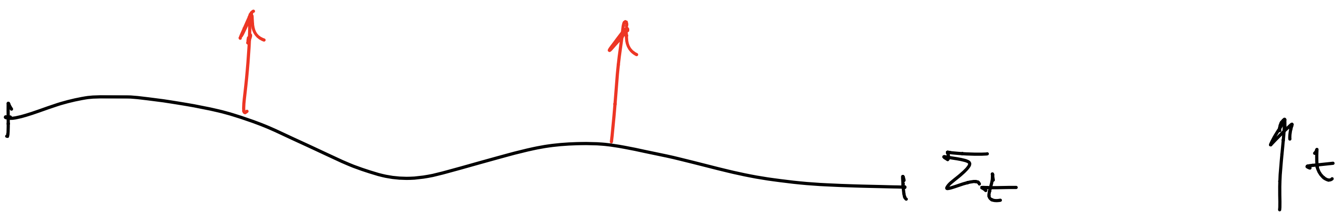
⋮

3 + 1 splitting



Ignore that space and time are equivalent and select t = const hypersurfaces.





Introduce a one-form $\Omega_\mu := \nabla_\mu t$

Ω_μ: sets the direction in which t is varying.

Note that $\Omega_\mu \Omega^\mu \neq 1$

$$|\Omega|^2 = g^{\mu\nu} \nabla_\mu t \nabla_\nu t = f(x^i, t) =: -\alpha^{-2}$$

lapse function

$\alpha > 0$: only constraint

$\underline{n}$: unit timelike normal vector ; $\underline{n} \cdot \underline{n} = -1$

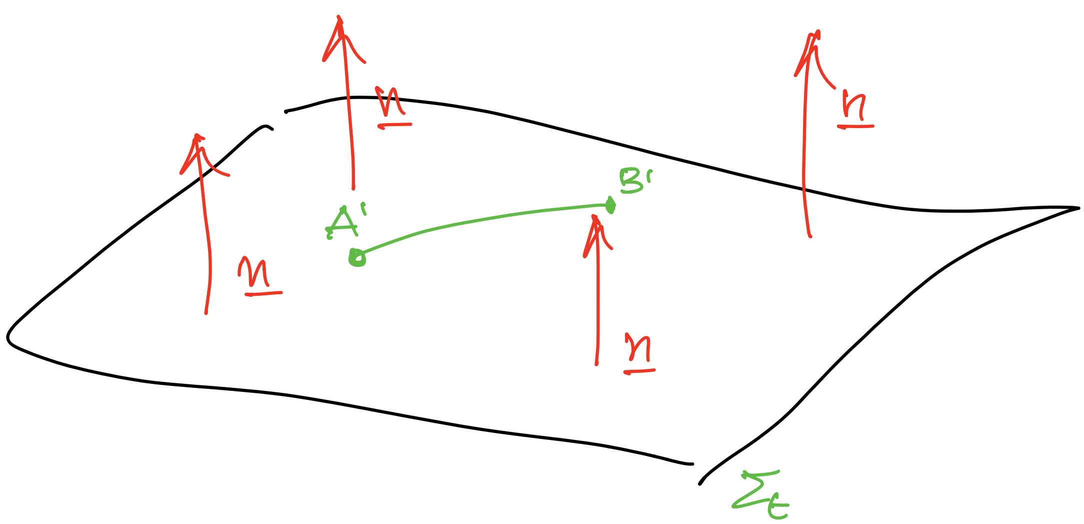
$\underline{n}$, $\underline{\Omega}$?

$$n^\mu = -\alpha g^{\mu\nu} \Omega_\nu = -\alpha g^{\mu\nu} \nabla_\nu t$$

now I can enforce $\underline{n} \cdot \underline{n} = -1$

$$n_\mu n^\mu = -\alpha^2 g^{\mu\nu} \Omega_\nu \Omega_\mu = -\alpha^2 \cdot \Omega_\mu \Omega^\mu$$

$$= -\alpha^2 \cdot (-\alpha^{-2}) = 1$$

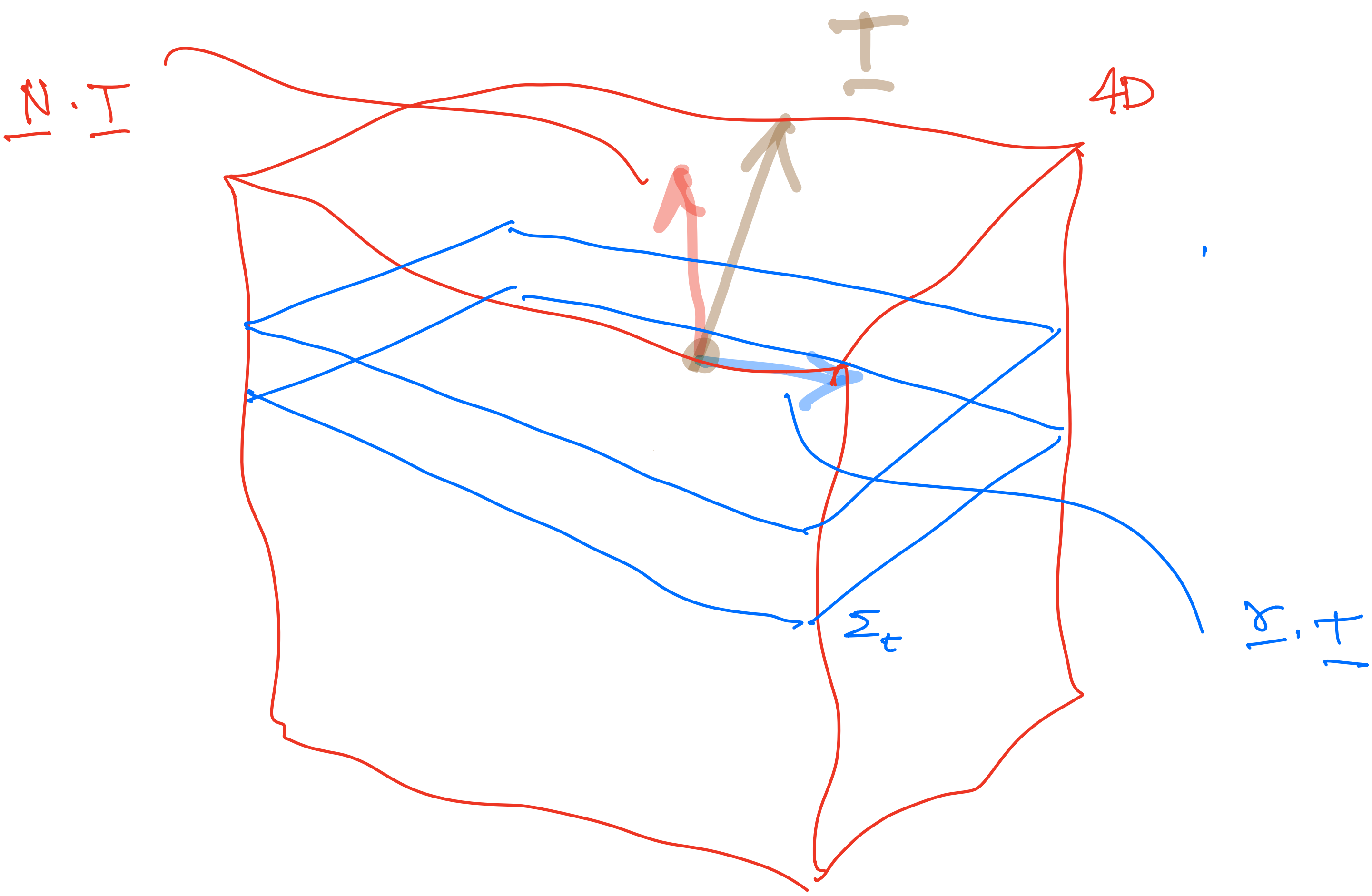


We can now introduce the concept of the spatial metric.

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$\gamma_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu \qquad : \text{ spatial 3 metric}$$

Use $\underline{\gamma}$ to measure distances on Σ_t



Define a time projector $\underline{N}$

$$N^{\mu}{}_{\nu} := -n^{\mu} n_{\nu}$$

$$\underline{\gamma} \cdot \underline{N} = 0$$

Q. How do I define a totally spatial covariant derivative?

$$\underbrace{D_{\alpha} T^{\beta}{}_{\gamma}}_{\text{3D cov. deriv.}} = \gamma^{\rho}{}_{\alpha} \gamma^{\beta}{}_{\sigma} \gamma^{\tau}{}_{\gamma} \underbrace{\nabla_{\rho} T^{\sigma}{}_{\tau}}_{\text{4D cov. derivative}}$$

Note

$$D_{\alpha} \gamma^{\alpha\beta} = 0$$

3D cov. deriv. is compatible with the metric

We need to "export" the concepts of curvature in 4D to a 3D spatial slice

$$(\underline{g} \to \underline{\gamma})$$

$$\Gamma^{\alpha}_{\mu\nu} \Longrightarrow {}^{(3)}\Gamma^{\alpha}_{\mu\nu} =$$

$$= \frac{1}{2}\gamma^{\alpha\delta}\left(\partial_\nu \gamma_{\delta\mu} + \partial_\mu \gamma_{\nu\delta} - \partial_\delta \gamma_{\mu\nu}\right)$$

$${}^{(3)}\Gamma \Longrightarrow \underbrace{{}^{(3)}R^{\mu}{}_{\alpha\mu\beta}}_{\text{3D Riemann}} \longrightarrow \underbrace{{}^{(3)}R_{\alpha\beta}}_{\text{3D Ricci}}$$

$${}^{(3)}R = {}^{(3)}R^{\mu}{}_{\mu}$$

$${}^{(3)}R^{\mu}{}_{\delta\alpha\beta}\, n_\mu = 0$$

$$2 D_{[\alpha} D_{\beta]} W_\delta = \underbrace{{}^{(3)}R^{\mu}{}_{\delta\alpha\beta}} W_\mu$$

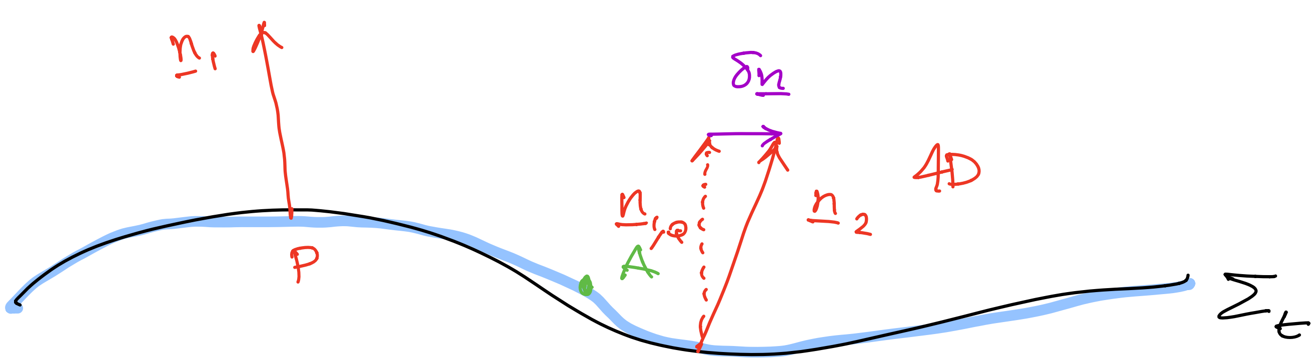
$$T_{[\alpha\beta]} = \frac{1}{2}\left(T_{\alpha\beta} - T_{\beta\alpha}\right)$$

$$R^{\alpha\mu\nu\beta} \rightsquigarrow {}^{(3)}R^{\alpha\mu\nu\beta} \quad : \text{intrinsic curvature}$$

4D 3D

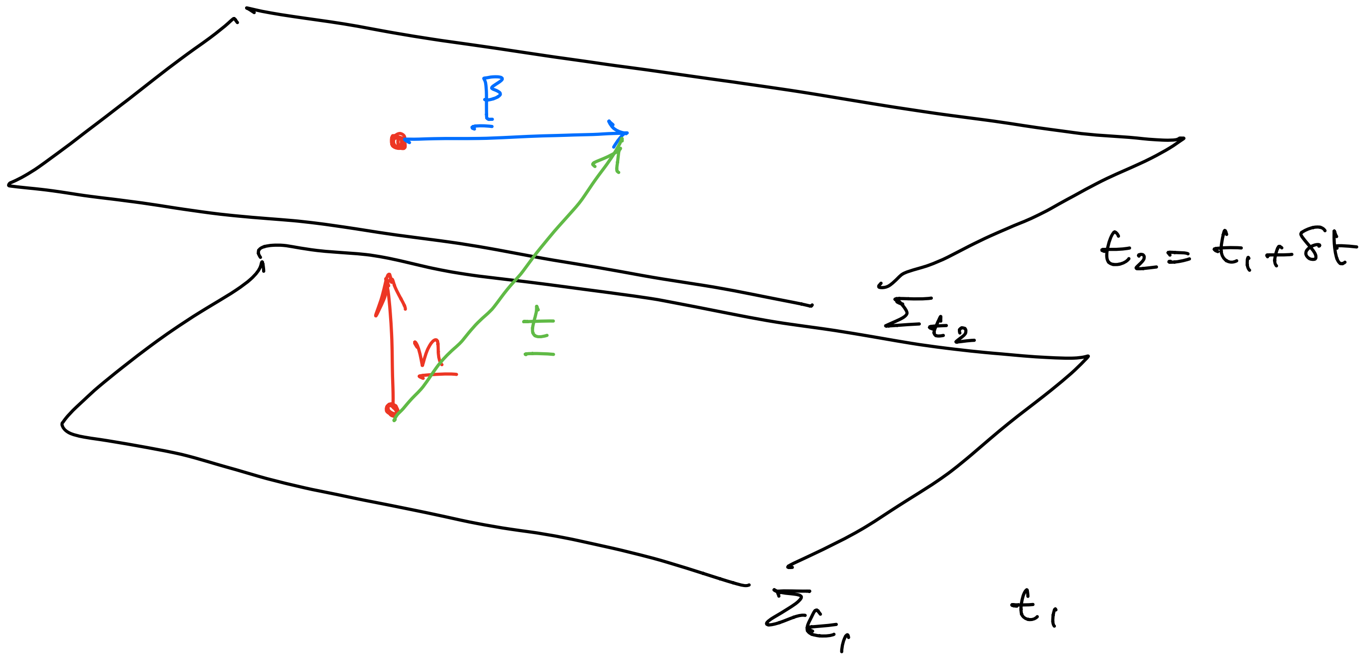
Where is the missing information ?

The missing information is called the extrinsic curvature and tells us how Σ_t is embedded in the 4D $\mathcal{M}$



$$K_{\mu\nu} := -\gamma^{\lambda}_{\mu} \, D_\lambda n_\nu \quad : \text{extrinsic curvature}$$

sym. tensor of rank 2 : 6 components.



$\underline{t} = \underline{n}$: bad choice as it does not take into account that coords can change between t_1 and t_2

$\underline{t} = \underline{n} + \underline{\beta}$

$\underline{\beta}$: fully spatial 4-vector

Why do I need a shift vector?

This is because n and Ω are not dual to each other.

$$n^\mu \Omega_\mu = n^\mu \nabla_\mu t = -\alpha\, \Omega^\mu \Omega_\mu = \frac{1}{\alpha} \neq 1$$

In other words if I use $\underline{n}$ I have no idea of where I will land on Σ_{t_2}

Because of this, I need two corrections:

1. normalizing factor in front of $\underline{n}$
2. spatial shift vector.

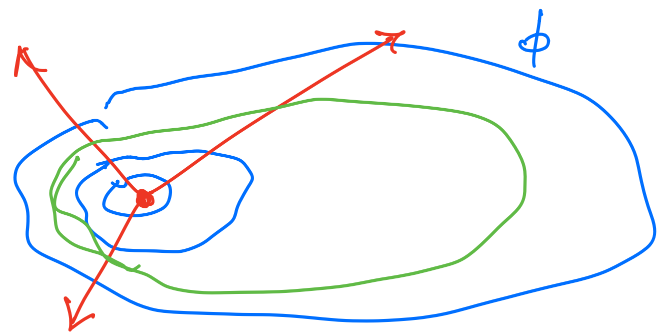
Imagine you have a scalar function

$$\phi \;:\; \Omega_\mu = \nabla_\mu \phi$$

$$\underline{t} \cdot \underline{\Omega} = \text{const}$$ and not a function

if $\underline{t} = \underline{n}$ $\qquad$ $\underline{t} \cdot \underline{\Omega} = \dfrac{1}{\alpha}$: function

and so we are not guaranteed that changes of ϕ in the direction $\underline{t}$ are all the same.



$\Omega_\mu = \nabla_\mu \phi$

$\underline{n} \cdot \underline{\Omega} = \text{const}$

In order to compensate for the fact that $\underline{n} \cdot \underline{\Omega} \neq \text{const}$

$\underline{t} = \alpha \underline{n}$

α: guarantees that all events on Σ_{t_1} will also be transported onto Σ_{t_2}

$$\underline{t} \cdot \underline{\Omega} = \alpha \underline{n} \cdot \underline{\Omega} = \alpha \cdot \frac{1}{\alpha} = 1 \quad \checkmark$$

In addition, the whole map may be moving (eg on a coord. system in the moon) so that you need to take that into account.

$$\boxed{\underline{t} = \alpha \underline{n} + \underline{\beta}}$$

We are ready now to have a look at the form of the metric in the 3+1 split

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$= -(\alpha^2 - \beta^i \beta_i)\, dt^2 + 2\beta_i\, dt\, dx^i + \gamma_{ij}\, dx^i dx^j$$

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{pmatrix}$$

$$ds^2 = -(\alpha^2 - \beta^i\beta_i)\,dt^2 + 2\beta_i\,dt\,dx^i + \gamma_{ij}\,dx^i dx^j$$

α: lapse

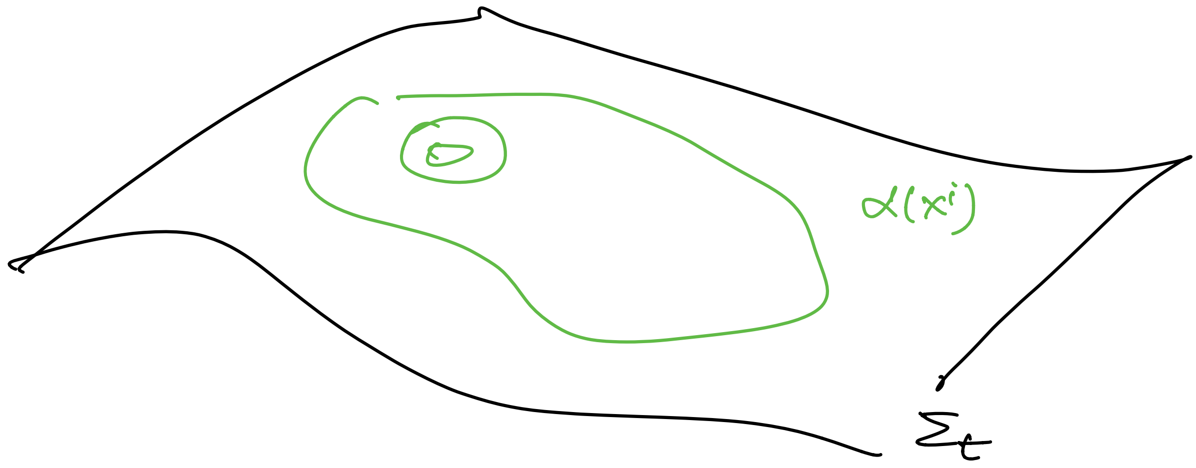
what does the lapse tell us?

$dx^i = 0 = \beta^i$

Coord. time

$$ds^2 = -\alpha^2 dt^2 = -d\tau^2 \implies \boxed{d\tau = \alpha\, dt}$$

proper time



$\alpha \in [0,1]$

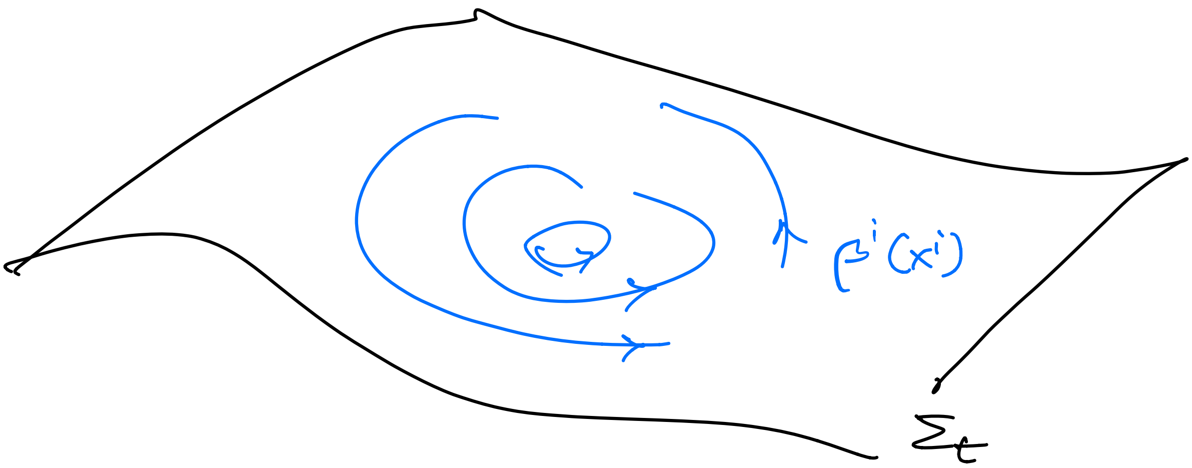
$\alpha = 1$: flat ~~space~~ spacetime

$\alpha \ll 1$: strong curvature region $dt \to 0$

what is the shift ?

$$x^i_{t_2} = x^i_{t_1} + \beta^i \, dt$$

the shift informs us about the changes in coords between Σ_{t_1} and Σ_{t_2}



If the spacetime intrinsically introduces a change in the coords, the shift can be used to minimize the coordinate distortion.

So far we have not discussed at all the Einstein equations. All we discussed is differential geometry in 3+1 split.

However, a lot of the groundwork has already been done.

Steps necessary:

$$^{(3)}R_{\alpha\beta\gamma\delta} + K_{\alpha\gamma}K_{\beta\delta} - K_{\alpha\delta}K_{\beta\gamma} = \gamma^{\mu}_{\alpha}\gamma^{\nu}_{\beta}\gamma^{\sigma}_{\delta}\gamma^{\rho}_{\gamma}R_{\mu\nu\sigma\rho}$$

(Gauss eqs.)

$$\boxed{\gamma\cdot\gamma\cdot\gamma\cdot\gamma\cdot \text{Riem}}$$

$$D_{\alpha}K_{\beta\gamma} - D_{\beta}K_{\alpha\gamma} = \gamma^{\rho}_{\beta}\gamma^{\mu}_{\alpha}\gamma^{\nu}_{\gamma}n^{\sigma}R_{\mu\nu\sigma\rho}$$

(Codazzi eqs.)

$$\boxed{\gamma\cdot\gamma\cdot\gamma\cdot n\cdot \text{Riem}}$$

$$\mathcal{L}_{\underline{n}} K_{\alpha\beta} = n\, n\, \gamma\, \gamma\, R_{\nu\delta\mu\gamma} - \frac{1}{2} D_{\alpha}D_{\beta}\alpha - K^{\gamma}_{\beta}K_{\alpha\gamma}$$

$$\boxed{\gamma\cdot\gamma\cdot n\cdot n\, \text{Riem}}$$

(Ricci equations)

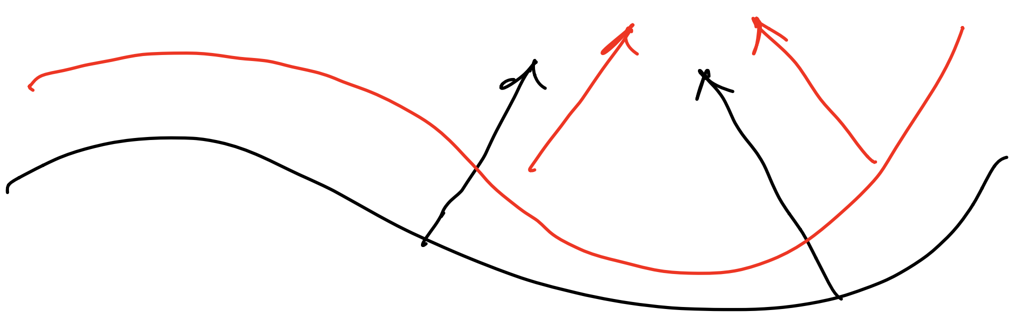
After a bit more algebra, where we do the same decomposition ($\gamma\gamma \ldots$) on the en. mom. tensor, we obtain the following eqs.

$$(*)\quad \partial_t K_{ij} = -D_i D_j \alpha + \alpha \left(R_{ij} - 2K_{ik}K^{k}{}_{j} + K K_{ij}\right)$$

$$- 8\pi\alpha\left(R_{ij} - \frac{1}{2}\gamma_{ij}\,(S - e)\right) + \mathcal{L}_{\underline{\beta}} K_{ij}$$

$$(**)\quad \partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_{\underline{\beta}} \gamma_{ij} \qquad K := K^{i}{}_{i}$$

ADM evolution equations.



$\underline{\gamma}$: position ; $\underline{K} \sim \partial_t \underline{\gamma}$ ~ velocity

$\underline{R}$, e, k_{ij} nonzero will induce a change

in K_{ij} $\Rightarrow$ $\underline{R}$, e, ... <u>acceleration</u>

$\underline{T}$: $T_{\mu\nu} = (e+p) u_\mu u_\nu + P g_{\mu\nu}$

$e := n^\mu n^\nu T_{\mu\nu}$: energy density

$S_{\mu\nu} = \gamma_\mu^{\ \alpha} \gamma_\nu^{\ \beta} T_{\alpha\beta}$

$S := S^\mu_{\ \mu}$

$\mathcal{L}_{\underline{n}}$: Lie derivative along normal

orth. coord system

$\underline{t} = \alpha \underline{n} + \beta$

$\alpha \mathcal{L}_{\underline{n}} - \mathcal{L}_{\underline{\beta}} = \mathcal{L}_{\underline{t}} = \partial_t$

(*), (**) are : 6 + 6 = 12 eqs. 1st-order

How many eqs are the Einstein eq?

$G_{\mu\nu} = 8\pi T_{\mu\nu}$: 10 eqs 2nd-order

$\Downarrow$

20 eqs in 1st order

∂_t^2 ; ∂_t

Missing 8 eqs.

There are other projections that one can do of the Riemann tensor

$$2 n^\mu n^\nu G_{\mu\nu} = {}^{(3)}R + K^2 - K_{\mu\nu} K^{\mu\nu} =$$

$$= 16\pi e$$

(Hamiltonian constraint)

$$-\gamma^{\mu}{}_{\alpha}\, n^{\nu}\, G_{\mu\nu} = -{}^{(3)}R_{\alpha\nu}\, n^{\nu} + \frac{1}{2} n_{\alpha} R$$

from which one obtains

$$D_{\nu} K^{\nu}{}_{\mu} - D_{\mu} K = 8\pi j_{\mu}$$

$j_{\mu} = -\gamma^{\alpha}{}_{\mu}\, n^{\beta}\, T_{\alpha\beta}$: mom. density

momentum constraints

$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha \left(R_{ij} - 2K_{ik}K^{k}{}_{j} + K K_{ij}\right)$$

(6)

$$-8\pi\alpha\left(R_{ij} - \frac{1}{2}\gamma_{ij}\,(S-e)\right) + \mathcal{L}_{\underline{\beta}} K_{ij}$$

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_{\underline{\beta}}\, \gamma_{ij} \qquad K := K^{i}{}_{i}$$

(6)

$${}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu} = 16\pi e$$

(1)

ADM eqs.

$$D_{\nu} K^{\nu}{}_{\mu} - D_{\mu} K = 8\pi j_{\mu}$$

(3)

16 eqs.

$\partial_t \alpha = \dots \quad ; \quad \partial_t \beta^i = \dots$

Still missing 4 eqs but this is fine because GR is a covariant theory and the missing equations are those for the evolution of the 4 unspecified degrees of freedom: α, β^i $\square$

Unfortunately, the ADM eqs. are only <u>weekly hyperbolic</u>

Einstein eqs

$$\partial_t \underline{U} + \underline{A} \nabla \underline{F} = S$$

$$\gamma_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \boxed{\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}}_{\gamma_{ij}} & \\ 0 & & & \end{pmatrix}$$

A weekly hyperbolic system is not necessarily well-posed (hyperbolic $\Rightarrow$ well-posedness) and so the solution ~~my~~ will! diverge

The blue eqs. do not contain time derivatives and are called constraint eqs. Think of EM

$$\partial_t E_i = \epsilon_{ijk} D^j B^k - 4\pi J_i$$

$$\partial_t B_i = - \epsilon_{ijk} D^j E^k$$

evolution eqs

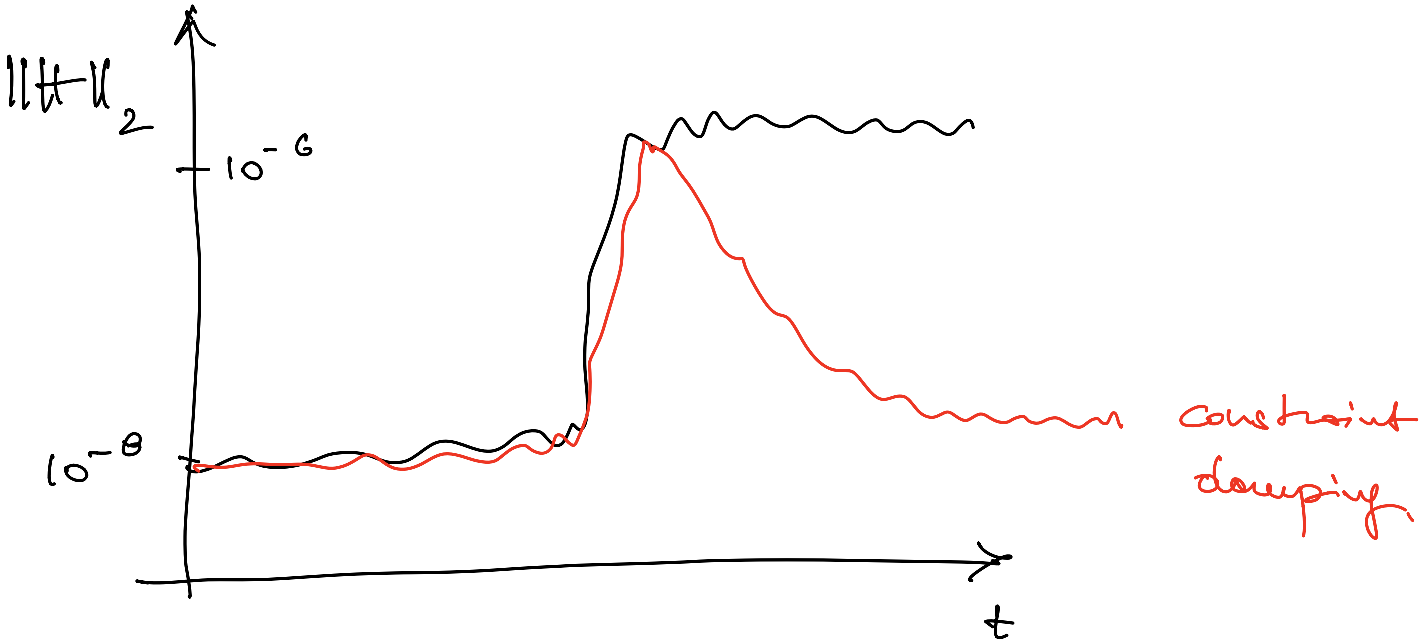
$$\partial_i E^i = 4\pi \rho_e$$

$$\partial_i B^i = 0$$

constraint eqs.

If $\partial_i B^i = 0$ at $t=0$, $\partial_i B^i = 0 \ \forall\, t > 0$

In practice we don't solve for the constraint eqs but monitor them



The ADM eqs are now replaced by a new set of eqs that are very similar but introduce new fields such that the system is hyperbolic, hence well posed.

BSSNOK formulation

CCZ4 " : constraint damping

FO - CCZ4 "