The **maths** of binary neutron star mergers

Lecture I

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Plan of the lectures

- ✴Lecture I: the **math** of neutron-star mergers
- ✴Lecture II: the **physics** of neutron-star mergers
- ✴Lecture III: the **astrophysics** of neutron-star mergers

✴Alcubierre, *"Introduction to 3+1 Numerical Relativity"*, Oxford University Press, 2008

✴Baumgarte and Shapiro, *"Numerical Relativity: Solving Einstein's Equations on the Computer"*, Cambridge University Press, 2010

✴Gourgoulhon, *"3+1 Formalism in General Relativity"*, Lecture Notes in Physics, Springer 2012

✴**Rezzolla and Zanotti, "Relativistic Hydrodynamics", Oxford University Press, 2013**

The equations of numerical relativity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}\,, \qquad \text{(field equations)}$$

$$\nabla_\mu T^{\mu\nu} = 0\,, \qquad \text{(cons. energy/momentum)}$$

$$\nabla_\mu (\rho u^\mu) = 0\,, \qquad \text{(cons. rest mass)}$$

$$p = p(\rho, \epsilon, Y_e, \ldots)\,, \qquad \text{(equation of state)}$$

$$\nabla_\nu F^{\mu\nu} = I^\mu\,, \quad \nabla_\nu^* F^{\mu\nu} = 0\,, \qquad \text{(Maxwell equations)}$$

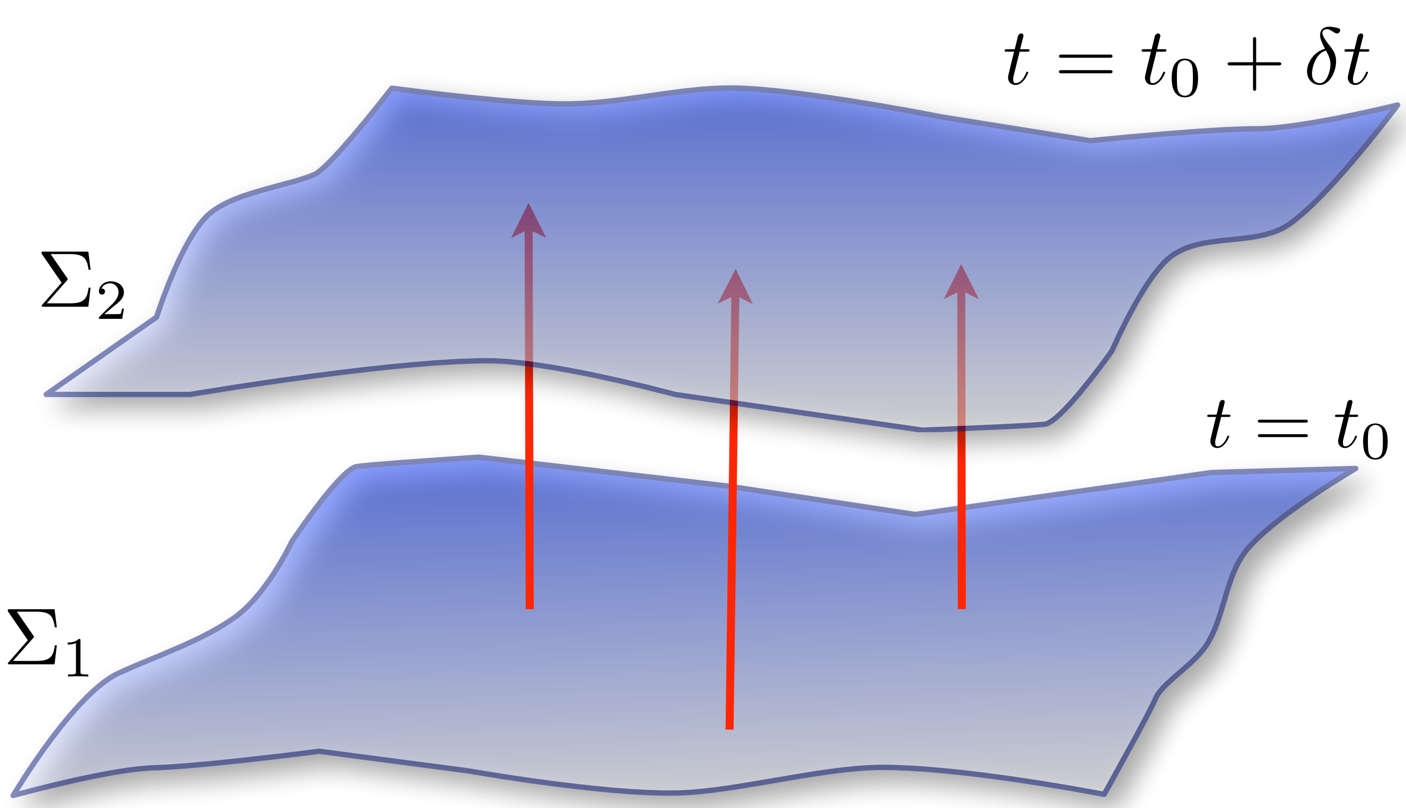
$$T_{\mu\nu} = T_{\mu\nu}^{\text{fluid}} + T_{\mu\nu}^{\text{EM}} + \ldots \qquad \text{(energy − momentum tensor)}$$

In GR these equations do not possess an analytic solution in the nonlinear regimes we are interested in

3+1 splitting of spacetime

First step: foliate the 4D spacetime

Given a manifold $\mathcal{M}$ describing a spacetime with 4-metric $g_{\mu\nu}$ we want to foliate it via spacelike, three-dimensional hypersurfaces, i.e., $\Sigma_1, \Sigma_2, \ldots$ leveled by a scalar function. The time coordinate t is an obvious good choice.



Define therefore

$$\Omega_\mu \equiv \nabla_\mu t$$

such that

$$|\Omega|^2 \equiv g^{\mu\nu} \nabla_\mu t \nabla_\nu t = -\alpha^{-2}$$

This defines the "lapse" function which is strictly positive for spacelike hypersurfaces

$$\alpha(t, x^i) > 0$$

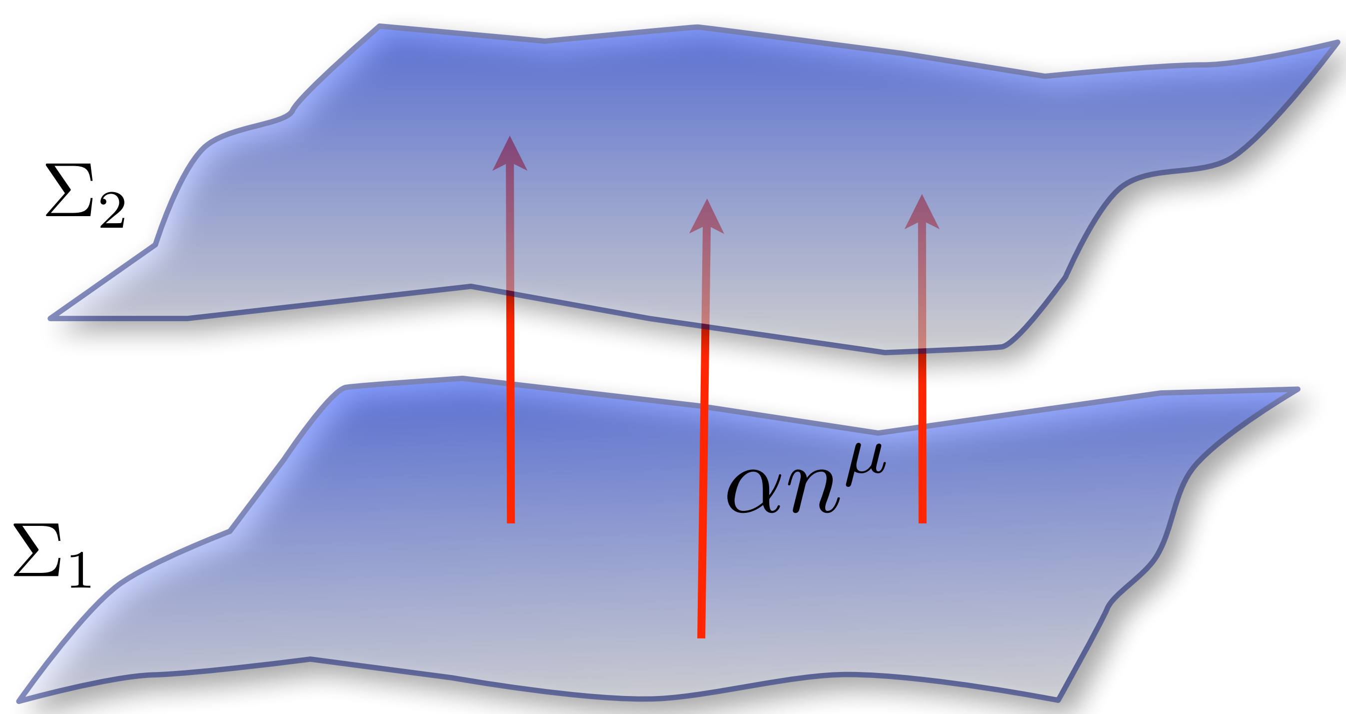
The lapse function allows then to do two important things:

i) define the unit normal vector to the hypersurface Σ

$$n^\mu \equiv -\alpha g^{\mu\nu}\Omega_\nu = -\alpha g^{\mu\nu}\nabla_\nu t$$

where

$$n^\mu n_\mu = -1$$



ii) define the spatial metric

$$\gamma_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$$

Second step: decompose 4-dim tensors

$\boldsymbol{n}$ and $\boldsymbol{\gamma}$ provide two useful tools to decompose any 4-dim. tensor into a purely spatial part (hence in Σ) and a purely timelike part (hence orthogonal to Σ and aligned with $\boldsymbol{n}$).

The spatial part is obtained after contracting with the spatial projection operator

$$\gamma^{\mu}{}_{\nu} = g^{\mu\alpha}\gamma_{\alpha\nu} = g^{\mu}{}_{\nu} + n^{\mu}n_{\nu} = \delta^{\mu}{}_{\nu} + n^{\mu}n_{\nu}$$

while the timelike part is obtained after contracting with the timelike projection operator

$$N^{\mu}{}_{\nu} = -n^{\mu}n_{\nu}$$

where the two projectors are obviously orthogonal

$$\gamma^{\nu}{}_{\mu}N^{\mu}{}_{\nu} = 0$$

It is now possible to define the 3-dim covariant derivative of a spatial tensor. This is simply the projection on Σ of all the indices of the the 4-dim. covariant derivative

$$D_\alpha T^\beta{}_\delta = \gamma^\rho{}_\alpha \gamma^\beta{}_\sigma \gamma^\tau{}_\delta \nabla_\rho T^\sigma{}_\tau$$

which, as expected, is compatible with spatial metric

$$D_\alpha \gamma^\beta{}_\delta = 0$$

All of the 4-dim tensor algebra can be extended straightforwardly to the 3-dim. spatial slice, so that the 3-dim covariant derivative can be expressed in terms of the 3-dimensional connection coefficients:

$${}^{(3)}\Gamma^\alpha{}_{\beta\delta} = \frac{1}{2}\gamma^{\alpha\mu}(\gamma_{\mu\beta,\delta} + \gamma_{\mu\delta,\beta} - \gamma_{\beta\delta\,\mu})$$

Similarly, the 3-dim Riemann tensor associated with $\boldsymbol{\gamma}$ is defined via the double 3-dimensional covariant derivative of any spatial vector $\boldsymbol{W}$, ie

$$2D_{[\alpha}D_{\beta]}W_\delta = {}^{(3)}R^\mu{}_{\delta\alpha\beta}W_\mu$$

where

$${}^{(3)}R^\mu{}_{\delta\alpha\beta}n_\mu = 0 \qquad \text{and} \qquad 2T_{[\alpha\beta]} = T_{\alpha\beta} - T_{\beta\alpha}$$

More explicitly, the 3-dim Riemann tensor can be written in terms of the 3-dim connection coefficients as

$${}^{(3)}R^\alpha{}_{\beta\gamma\delta} = {}^{(3)}\Gamma^\alpha{}_{\beta\delta,\gamma} - {}^{(3)}\Gamma^\alpha{}_{\beta\gamma,\delta} + {}^{(3)}\Gamma^\mu{}_{\beta\delta}{}^{(3)}\Gamma^\alpha{}_{\mu\gamma} - {}^{(3)}\Gamma^\mu{}_{\beta\gamma}{}^{(3)}\Gamma^\alpha{}_{\mu\delta}$$

Also, the 3-dim contractions of the 3-dim Riemann tensor, i.e. the 3-dim Ricci tensor the 3-dim Ricci scalar are respectively given by

$${}^{(3)}R_{\alpha\beta} = {}^{(3)}R^\delta{}_{\alpha\delta\beta} \qquad\qquad {}^{(3)}R = {}^{(3)}R^\delta_\delta$$

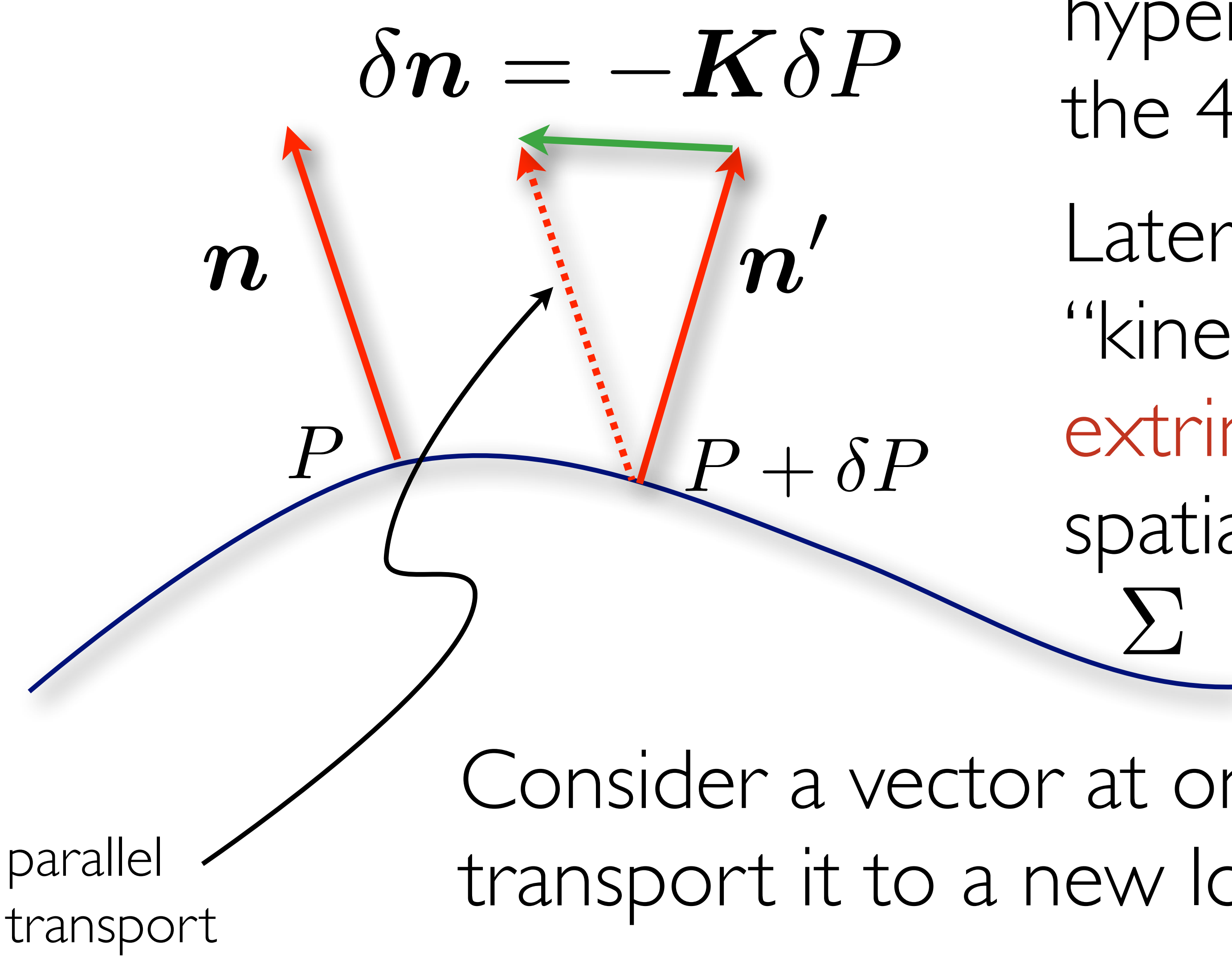
It is important not to confuse the 3-dim Riemann tensor ${}^{(3)}R^{\mu}{}_{\delta\alpha\beta}$ with the corresponding 4-dim one $R^{\mu}{}_{\delta\alpha\beta}$

${}^{(3)}R^{\mu}{}_{\delta\alpha\beta}$ is a 4-dimensional tensor but it is purely spatial (spatial derivatives of spatial metric $\boldsymbol{\gamma}$)

$R^{\mu}{}_{\delta\alpha\beta}$ is a full 4-dimensional tensor containing also time derivatives of the full 4-dim metric $\boldsymbol{g}$

The information present in $R^{\mu}{}_{\delta\alpha\beta}$ and "missing" in ${}^{(3)}R^{\mu}{}_{\delta\alpha\beta}$ can be found in another spatial tensor: the extrinsic curvature.
As we shall see, this information is indeed describing the time evolution of the spatial metric

More geometrically, the extrinsic curvature measures the changes in the normal vector under parallel transport



Hence it measures how the 3-dim hypersurface is "bent" with respect to the 4-dim spacetime

Later on we will discuss also a "kinematical" interpretation of the extrinsic curvature in terms of the spatial metric

Consider a vector at one position P and parallel-transport it to a new location $P + \delta P$

The difference in the two vectors is proportional to the extrinsic curvature and this can be positive or negative

$$K_{\mu\nu} := -\gamma^{\lambda}{}_{\mu} D_{\lambda} n_{\nu}$$

Since the extrinsic curvature measures the bending of the spacelike hypersurface, two more equivalent definitions exists for the extrinsic curvature:

2) in terms of the acceleration of normal observers:

$$K_{\mu\nu} := -D_\mu n_\nu - n_\mu a_\nu = -D_\mu n_\nu - n_\mu n^\lambda D_\lambda n_\nu$$

3) in terms of the Lie derivative of the spatial metric:

$$K_{\mu\nu} := -\frac{1}{2}\mathcal{L}_{\boldsymbol{n}}\gamma_{\mu\nu}$$

1) in terms of the Lie derivative of the spatial metric:

$$K_{\mu\nu} := -\gamma^\lambda{}_\mu D_\lambda n_\nu$$

Finding a direction for evolutions

Note that the unit normal $\boldsymbol{n}$ to a spacelike hypersurface Σ is not the natural time derivative. This is because $\boldsymbol{n}$ is not dual to the surface 1-norm $\boldsymbol{\Omega}$, i.e.

$$n^\mu \Omega_\mu = n^\mu \nabla_\mu t = -\alpha \Omega^\mu \Omega_\mu = \frac{1}{\alpha}$$

We need therefore to find a new vector along which to carry out the time evolutions and that is dual to the surface 1-norm. Such a vector is easily defined as

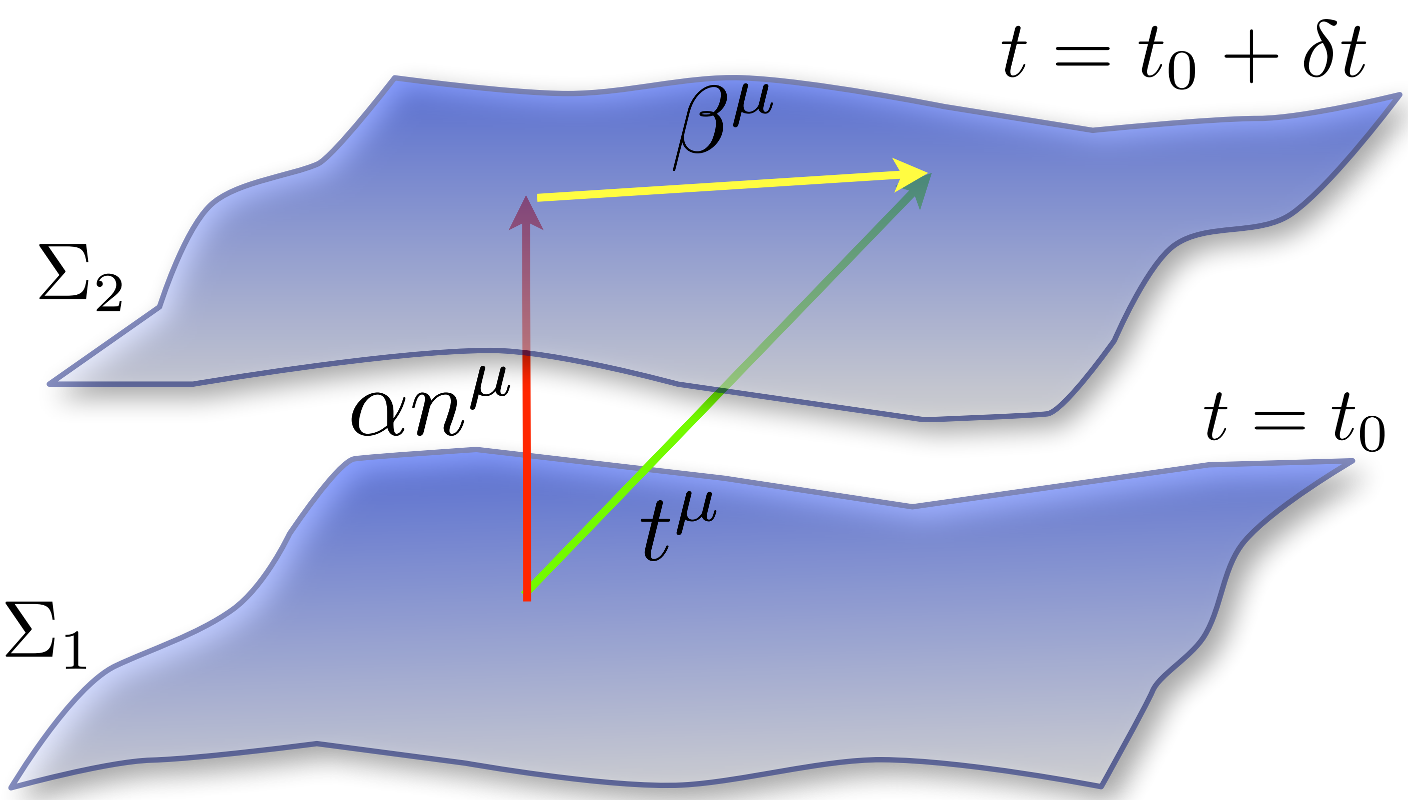
$$t^\mu \equiv \alpha n^\mu + \beta^\mu$$

where $\boldsymbol{\beta}$ is any spatial "shift" vector.

Clearly now the two tensors are dual to each other, ie

$$t^\mu \Omega_\mu = \alpha n^\mu \Omega_\mu + \beta^\mu \Omega_\mu = \alpha / \alpha = 1$$

Because the vector t^μ is dual to the 1-form Ω_μ , we are guaranteed that the integral curves of t^μ are naturally parametrized by the time coordinate.



Stated differently, all infinitesimal vectors t^μ originating on one hypersurface Σ_1 would end up on the same hypersurface Σ_2

This is not guaranteed for translations along Ω^μ

A more intuitive description of the lapse function α and of the shift vector β^μ will be presented once we introduce a coordinate basis

Note that t^μ is not necessarily timelike if the shift is superluminal

$$t^\mu t_\mu = -\alpha^2 + \beta^\mu \beta_\mu \lesseqgtr 0$$

With this definition we can revise the Lie derivative along the unit normal $\mathcal{L}_{\boldsymbol{n}}$. Since

$$\alpha \mathcal{L}_{\boldsymbol{n}} = \mathcal{L}_{\boldsymbol{t}} - \mathcal{L}_{\boldsymbol{\beta}}$$

the Ricci equation we have encountered before: $K_{\alpha\beta} = -\frac{1}{2}\mathcal{L}_{\boldsymbol{n}}\gamma_{\alpha\beta}$ can now be rewritten as

$$\mathcal{L}_{\boldsymbol{t}}\gamma_{\mu\nu} = -2\alpha K_{\mu\nu} + \mathcal{L}_{\boldsymbol{\beta}}\gamma_{\mu\nu} \qquad (*)$$

Once again, this a clear expression that the extrinsic curvature can be seen as the rate of change of the spatial metric, i.e.

$$K_{\mu\nu} \propto -\frac{1}{\alpha}\mathcal{L}_{\boldsymbol{t}}\gamma_{\mu\nu}$$

Finally, note that the Ricci equations (*) are definitions and not pieces of the Einstein eqs, although this is sometimes confused

Selecting a coordinate basis

So far we have dealt with tensor eqs and not specified a coordinate basis with unit vectors $\boldsymbol{e}_j$. Doing so can be useful to simplify equations and to highlight the "spatial" nature of $\boldsymbol{\gamma}$ and $\boldsymbol{K}$

The choice in this case in very simple. We want:
i) three of them have to be purely spatial, i.e.

$$n_\mu (\boldsymbol{e}_j)^\mu = 0 \qquad e.g. \qquad (\boldsymbol{e}_1)^\mu = (0, 1, 0, 0)$$

ii) the fourth one has to be along the vector $\boldsymbol{t}$, i.e.

$$(\boldsymbol{e}_0)^\mu = t^\mu = (1, 0, 0, 0)$$

As a result:

$$\mathcal{L}_{\boldsymbol{t}} = \partial_t$$

i.e. the Lie derivative along $\boldsymbol{t}$ is a simple partial derivative

$$n_j = n_\mu (e_j)^\mu = 0 \quad \text{but} \quad n_0 \neq 0$$

i.e. the space covariant components of a **timelike** vector are zero; only the covariant time component survives

$$n_\mu \beta^\mu = \beta^0 n_0 = 0 \quad \Longrightarrow \quad \beta^0 = 0 \quad \Longrightarrow \quad \beta^\mu = (0, \beta^j)$$

i.e. the time contravariant component of a **spacelike** vector is zero; only the spatial contravariant components survive

Putting things together and bearing in mind that $n_\mu n^\mu = -1$

$$n^\mu = \frac{1}{\alpha}(1, -\beta^i)\,; \qquad n_\mu = (-\alpha, 0, 0, 0)$$

Because for any spatial tensor $T^{\mu 0} = 0$ the contravariant components of the metric in a 3+1 split are

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{pmatrix}$$

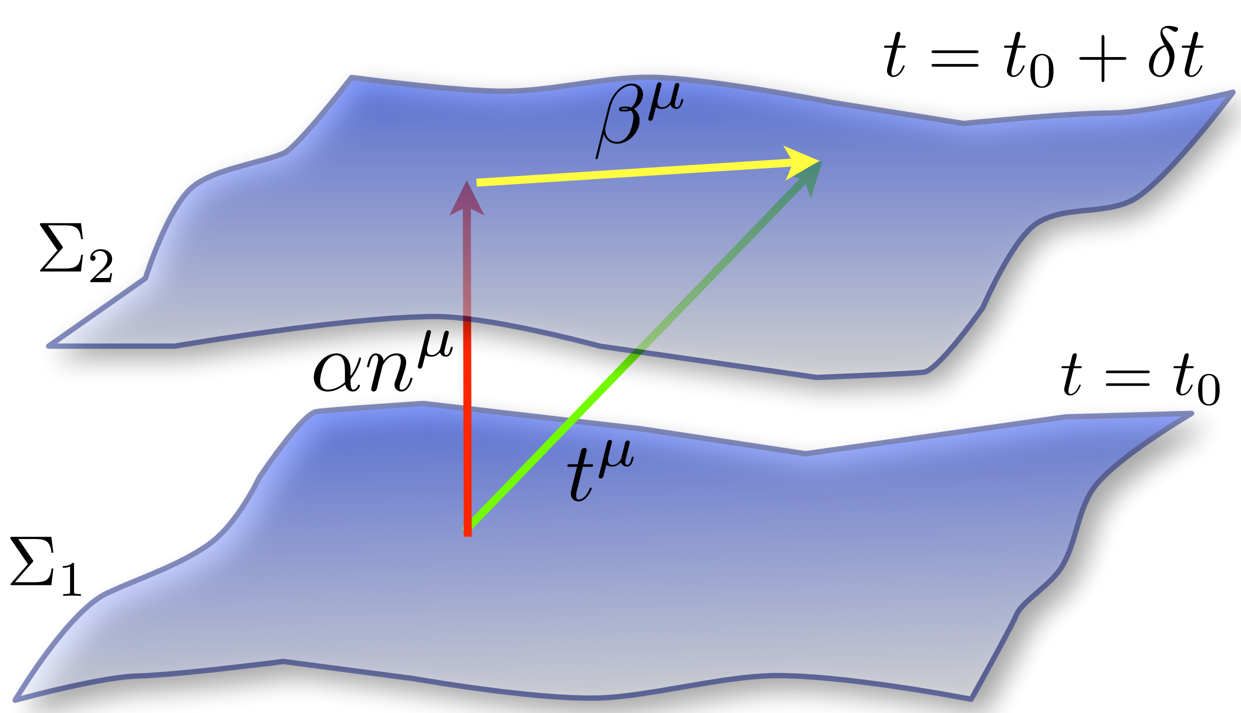
Similarly, since $g_{ij} = \gamma_{ij}$ the covariant components are

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

Note that $\gamma^{ik}\gamma_{kj} = \delta^i{}_j$ (i.e. $\gamma^{ij}, \ \gamma_{ij}$ are inverses) and thus they can be used to raise/lower the indices of **spatial** tensors

We can now have a more intuitive interpretation of the lapse, shift and spatial metric. Using the expression for the covariant 4-dim covariant metric, the line element is given

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(\alpha^2 - \beta^i\beta_i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij}dx^i dx^j$$



Hence:

the lapse measures proper time between two adjacent hypersurfaces

$$\mathrm{d}\tau^2 = -\alpha^2(t, x^j)\mathrm{d}t^2$$

the shift relates spatial coordinates between two adjacent hypersurfaces

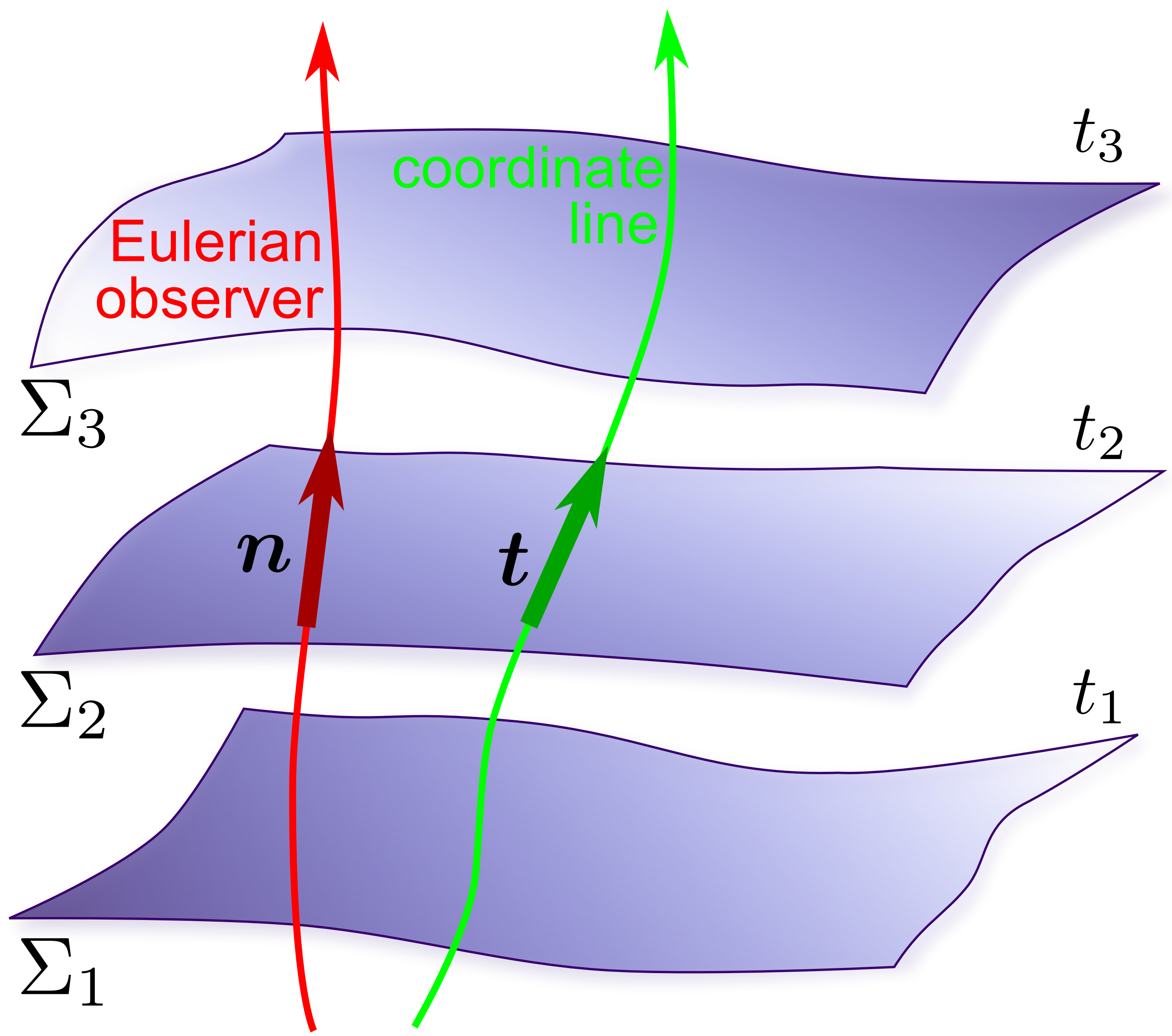
$$x^i_{t_0+\delta t} = x^i_{t_0} - \beta^i(t, x^j)\mathrm{d}t$$

the spatial metric measures distances between points on every hypersurface

$$\mathrm{d}l^2 = \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

We can now also distinguish between a normal line and a coordinate line.

Both are worldlines but the first one tells us about the evolution of the normal to the hypersurface, while the second one tells us about the evolution of a point in the coordinate chart



Decomposing the Einstein equations

- So far we have just played with differential geometry. No mention has been made of the Einstein equations.

- The 3+1 splitting naturally "splits" the Einstein equations into:
  - ✳ a set which is fully defined on each spatial hypersurfaces (and does not involve therefore time derivatives).
  - ✳ a set which instead relates quantities (i.e. spatial metric and extrinsic curvature) between two adjacent hypersurfaces.

- The first set is usually referred to as the "constraint" equations, while the second one as the "evolution" equation

Next, we need to decompose the Einstein equations in the spatial and timelike parts.

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}$$

and to do this we need to define a few identities

First we decompose the 4-dim Riemann tensor $R_{\alpha\beta\mu\nu}$ projecting all indices to obtain the Gauss equations

$$^{(3)}R_{\alpha\beta\gamma\delta} + K_{\alpha\gamma}K_{\beta\delta} - K_{\alpha\delta}K_{\beta\gamma} = \gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}\gamma^{\rho}{}_{\delta}\gamma^{\sigma}{}_{\gamma}\ R_{\mu\nu\sigma\rho}$$

Next, we make 3 spatial projections and a timelike one to obtain the Codazzi equations

$$D_{\alpha}K_{\beta\gamma} - D_{\beta}K_{\alpha\gamma} = \gamma^{\rho}{}_{\beta}\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\gamma}n^{\sigma}R_{\mu\nu\sigma\rho}$$

Finally we take 2 spatial projections and 2 timelike ones to obtain the Ricci equations

$$\mathcal{L}_{\boldsymbol{n}} K_{\alpha\beta} = n^\delta n^\gamma \gamma^\mu{}_\alpha \gamma^\nu{}_\beta R_{\nu\delta\mu\gamma} - \frac{1}{\alpha} D_\alpha D_\beta \alpha - K^\gamma{}_\beta K_{\alpha\gamma}$$

where the second derivative of the lapse has been introduced via the identity

$$a_\mu = D_\mu \ln \alpha$$

Another important identity which will be used in the following is

$$D_\mu U^\nu = \gamma_\mu{}^\rho \nabla_\rho U^\nu + K_{\mu\rho} U^\rho n^\nu$$

and which holds for any spatial vector $\boldsymbol{U}$ $(U^\mu n_\mu = 0)$

The RHS of the Einstein equations

We are now ready to express the missing piece of the 3+1 decomposition and derive the evolution part of the Einstein eqs.

We need suitable projections of the right-hand-side of the Einstein equations and in particular the two spatial ones, ie

$$\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}G_{\mu\nu} = 8\pi S_{\alpha\beta} \equiv 8\pi\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}T_{\mu\nu}$$

where the energy-momentum tensor of a perfect fluid is:

$$T_{\mu\nu} = (e+p)u_{\mu}u_{\nu} + pg_{\mu\nu} = h\rho u_{\mu}u_{\nu} + pg_{\mu\nu}$$

with

ρ : rest-mass density

p : pressure

ϵ : specific internal energy

$e = \rho(1+\epsilon)$: total energy density

$h = \dfrac{e+p}{\rho}$: specific enthalpy

$S \equiv S^{\mu}{}_{\mu}$

Since $n^\mu u_\mu = 1$, (the two vectors are parallel and unit vectors) the energy density measured by the normal observers will be given by the double timelike projection

$$e = n^\mu n^\nu T_{\mu\nu}$$

Similarly, the momentum density (i.e. the extension of the Newtonian mass current) will be given by the mixed time and spatial projection

$$j_\mu = -\gamma^\alpha{}_\mu n^\beta T_{\alpha\beta} = -(h\rho u_\mu + p n_\mu)$$

Just as a reminder, the fully spatial projection of the energy-momentum tensor was already introduced as

$$S_{\mu\nu} = \gamma^\alpha{}_\mu \gamma^\beta{}_\nu T_{\alpha\beta}$$

The (ADM) Einstein eqs in 3+1

We can write the Einstein equations in the 3+1 splitting of spacetime in a set of evolution and constraint equations as:

$$\boldsymbol{\gamma} \cdot \boldsymbol{\gamma} \cdot (\text{Einstein eqs}) \; + \; \text{Ricci eqs} \Longrightarrow$$

$$\begin{aligned} \partial_t K_{ij} = & -D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^{kj} + KK_{ij}) \\ & -8\pi\alpha(R_{ij} - \frac{1}{2}\gamma_{ij}(S - e)) + \mathcal{L}_{\boldsymbol{\beta}} K_{ij} \end{aligned}$$

[6 eqs]

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_{\boldsymbol{\beta}} \gamma_{ij}$$

[6 eqs]

These are known as the ADM (Arnowitt, Deser, Misner) eqs.:
12 weakly hyperbolic, first-order in time, second-order in space, nonlinear partial differential equations (evolution equations)

The constraint equations (I)

We first time-project twice the left-hand-side of the Einstein equations to obtain

$$2n^\mu n^\nu G_{\mu\nu} = {}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu}$$

Doing the same for the right-hand-side, using the Gauss eqs contracted twice with the spatial metric and the definition of the energy density we finally reach the form of the Hamiltonian constraint equation

$$ {}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu} = 16\pi e$$

Note that this is a single **elliptic equation** (hence not containing time derivative) which should be satisfied everywhere on the spatial hypersurface Σ

The constraint equations (II)

Similarly, with a mixed time-space projection of the left-hand-side of the Einstein equations we obtain

$$-\gamma^{\mu}{}_{\alpha} n^{\nu} G_{\mu\nu} = -{}^{(3)}R_{\alpha\nu} n^{\nu} + \frac{1}{2} n_{\alpha} R$$

Doing the same for the right-hand-side, using the contracted Codazzi equations and the definition of the momentum density we finally reach the form of the momentum constraint equations

$$D_{\nu} K^{\nu}{}_{\mu} - D_{\mu} K = 8\pi j_{\mu}$$

which are also 3 **elliptic equations**.

The 4 constraint equations are the necessary and sufficient integrability conditions for the embedding of the spacelike hypersurfaces $(\Sigma, \gamma_{\mu\nu}, K_{\mu\nu})$ in the 4-dim. spacetime $(\mathcal{M}, g_{\mu\nu})$

The (ADM) Einstein eqs in 3+1

Similarly

$$\boldsymbol{n} \cdot \boldsymbol{n} \cdot (\text{Einstein eqs}) \;+\; \text{Gauss eqs} \Longrightarrow$$

$$R + K^2 - K_{ij}K^{ij} = 16\pi e$$

Hamiltonian Constraint (HC) **[1 eq]**

$$\boldsymbol{\gamma} \cdot \boldsymbol{n} \cdot (\text{Einstein eqs}) \;+\; \text{Codazzi eqs} \Longrightarrow$$

$$D_j K^j{}_i - D_i K = 8\pi j_i$$

Momentum Constraints (MC) **[3 eqs]**

These are 1+3 elliptic (second-order in space), nonlinear partial differential equations: constraint equations

The (ADM) Einstein eqs in 3+1

All together we have:

$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^{kj} + KK_{ij}) - 8\pi\alpha(R_{ij} - \frac{1}{2}\gamma_{ij}(S - e)) + \mathcal{L}_{\boldsymbol{\beta}} K_{ij}$$ [6]

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_{\boldsymbol{\beta}} \gamma_{ij}$$ [6]

$$R + K^2 - K_{ij}K^{ij} = 16\pi e$$ [1]

$$D_j K^j{}_i - D_i K = 8\pi j_i$$ [3]

These are known as the **ADM** (Arnowitt, Deser, Misner) eqs.:
6+6 1st-order in time, 2nd-order in space hyperbolic eqs. (evolution eqs), **1+3 2nd-order elliptic eqs.** (constraint eqs.)

ADM vs Maxwell

The ADM eqs may appear as rather cryptic and simply complicated. However, it is easy to see analogies with the Maxwell eqs. and make the equations less cryptic.
The relevant quantities in this case are the electric and magnetic fields $\boldsymbol{E}, \boldsymbol{B}$, the charge density ρ_e and the charge current density $\boldsymbol{J}$

Then also the Maxwell equations split into evolution equations

$$\partial_t \boldsymbol{E} = \nabla \times \boldsymbol{B} - 4\pi \boldsymbol{J}\,, \quad \Longleftrightarrow \quad \partial_t E_i = \epsilon_{ijk} D^j B^k - 4\pi J_i\,,$$

$$\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}\,, \quad \Longleftrightarrow \quad \partial_t B_i = -\epsilon_{ijk} D^j E^k$$

and constraint equations

$$\nabla \cdot \boldsymbol{E} = 4\pi \rho_e\,, \quad \Longleftrightarrow \quad \partial_i E^i = 4\pi \rho_e\,,$$

$$\nabla \cdot \boldsymbol{B} = 0\,, \quad \Longleftrightarrow \quad \partial_i B^i = 0\,,$$

Also for the Maxwell eqs it is possible to show that **if** the constraints are satisfied initially, then the evolution eqs preserve this property. To further highlight the analogies let's introduce the vector potential

$$A_\mu = (\Phi, A_i)\,, \qquad \text{such that} \qquad B_i = \epsilon_{ijk} D^j A^k$$

and so that the Maxwell evolution equations become

$$\partial_t A_i = -E_i - D_i \Phi\,,$$

$$\partial_t E_i = -D^j D_j A_i + D_i D^j A_j - 4\pi J_i$$

to be compared with the ADM evolution eqs

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_{\boldsymbol{\beta}} \gamma_{ij}$$

$$\begin{aligned}\partial_t K_{ij} = & -D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^{kj} + KK_{ij}) \\ & -8\pi\alpha(R_{ij} - \frac{1}{2}\gamma_{ij}(S - e)) + \mathcal{L}_{\boldsymbol{\beta}} K_{ij}\end{aligned}$$

It is then possible to make the associations

$$\begin{aligned} \Phi \quad &\longleftrightarrow \quad \beta_i \\ A_i \quad &\longleftrightarrow \quad \gamma_{ij} \\ E_i \quad &\longleftrightarrow \quad K_{ij} \end{aligned}$$

and realize that the RHSs of the evolution equation of A_i/γ_{ij} involve a field variable E_i/K_{ij} and the spatial derivatives of a gauge quantity Φ/β_i

Similarly, the RHS of the evolution equation of E_i/K_{ij} involve matter sources as well as second spatial derivatives of the second field variable A_i/γ_{ij}

Indeed, the similarities between the ADM eqs and the Maxwell eqs written in terms of the vector potential (i.e. as in previous slide) are so large that they suffer of the same problems/ instabilities (more later)

In practice, the ADM are essentially never used!

These equations are perfectly alright mathematically but not in a form that is well suited for numerical implementation.

Indeed the system can be shown to be weakly hyperbolic and hence "ill-posed"

In practice, numerical instabilities rapidly appear that destroy the solution exponentially

However, the stability properties of numerical implementations can be improved by introducing certain new auxiliary functions and rewriting the ADM equations in terms of these functions.

How do we make the Maxwell eqs strongly hyperbolic?

i) use a specific gauge, e.g. Lorentz gauge: $\partial_t \Phi = -D^i A_i$ so that the eqs become simply

$$(\partial_t^2 - D^j D_j) A_i = \Box A_i = 4\pi J_i$$

This can be done also in GR by introducing harmonic coordinates and a generalized harmonic formulation of the Einstein eqs. (more later)

ii) implement a gauge-invariant approach by taking a time derivative of E_i rather than of A_i . This leads to

$$\partial_t^2 E_i = D_i D^j(-E_j - D_j \Phi) + D^j D_j(-E_i - D_i \Phi) - \partial_t J_i$$

which, using the constraint $D^i E_i = 4\pi \rho_e$ reduces to

$$\Box E_i = \partial_t J_i + 4\pi D_i \rho_e$$

This approach is aesthetically attractive but may lead to complications because the matter source term is proportional to spatial derivatives of the charge density $D_i \rho_e$. In GR this would correspond to derivatives of the rest-mass density and may be divergent if a shock is present

iii) introduce a new variable to remove the mix-derivative term, i.e. define $\Gamma \equiv D^i A_i$ so that the evolution eq. becomes

$$\partial_t E_i = -D^j D_j A_i + D_i \Gamma - 4\pi J_i$$

$$\Box A_i = -D_i \Gamma - D_i \partial_t \Phi + 4\,\pi J_i$$

Clearly an evolution equations is required for Γ which is a new variable like all the others (just no physical meaning):

$$\partial_t \Gamma = \partial_t D^i A_i = D^i \partial_t A_i = -D^i E_i - D_i D^i \Phi = -4\pi \rho_e - D_i D^i \Phi$$

We have gained one more eq. but the system is hyperbolic!

The same is done for the ADM eqs and new evolution variables are introduced to obtain a set of eqs that is strongly hyperbolic and hence well-posed (doesn't blow up).

$$\phi = \frac{1}{12}\ln(\det(\gamma_{ij})) = \frac{1}{12}\ln(\gamma),$$

$$\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij},$$

$$K = \gamma^{ij}K_{ij},$$

$$\tilde{A}_{ij} = e^{-4\phi}(K_{ij} - \frac{1}{3}\gamma_{ij}K),$$

$$\Gamma^i = \gamma^{jk}\Gamma^i_{jk}$$

$$\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}$$

ϕ: conformal factor

$\tilde{\gamma}_{ij}$: conformal 3-metric

K: trace of extrinsic curvature

$\tilde{A}_{ij}$: trace-free conformal extrinsic curvature

$\tilde{\Gamma}^i$:"Gammas"

are our new evolution variables

The ADM equations are then rewritten as

$$\mathcal{D}_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} \,, \qquad \text{where } \mathcal{D}_t \equiv \partial_t - \mathcal{L}_\beta$$

$$\mathcal{D}_t \phi = -\frac{1}{6}\alpha K \,,$$

$$\mathcal{D}_t \tilde{A}_{ij} = e^{-4\phi}\left[-\nabla_i \nabla_j \alpha + \alpha \left(R_{ij} - S_{ij}\right)\right]^{\mathrm{TF}} + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j\right) \,,$$

$$\mathcal{D}_t K = -\gamma^{ij}\nabla_i \nabla_j \alpha + \alpha\left[\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2 + \frac{1}{2}\left(\rho + S\right)\right] \,,$$

$$\mathcal{D}_t \tilde{\Gamma}^i = -2\tilde{A}^{ij}\partial_j \alpha + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K - \tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j \phi\right)$$
$$-\partial_j\left(\beta^l \partial_l \tilde{\gamma}^{ij} - 2\tilde{\gamma}^{m(j}\partial_m \beta^{i)} + \frac{2}{3}\tilde{\gamma}^{ij}\partial_l \beta^l\right) .$$

These equations are also known as the BSSNOK equations or more simply the conformal traceless formulation of the Einstein equations.

Although not self evident, the BSSNOK equations are strongly hyperbolic with a structure which is resembling the 1st-order in time, 2nd-order in space formulation

$$\Box\phi = 0 \quad \Longleftrightarrow \begin{cases} \partial_t \phi = \psi \\ \\ \partial_t \psi = \partial^i \partial_i \phi \end{cases}$$

scalar wave equation

$$\begin{cases} \partial_t \tilde{\gamma}_{ij} \propto \tilde{A}_{ij} \\ \\ \partial_t \tilde{A}_{ij} \propto D^i D_i \tilde{\gamma}_{ij} \end{cases}$$

BSSNOK

The BSSNOK is a widely used formulation of the Einstein eqs and used to simulate black holes and neutron stars. Other formulations have been recently suggested that have even better properties, e.g. CCZ4, Z4c.

Gauge conditions

The (ADM) Einstein eqs in 3+1

All together we have:

$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^{kj} + KK_{ij}) - 8\pi\alpha(R_{ij} - \frac{1}{2}\gamma_{ij}(S - e)) + \mathcal{L}_{\boldsymbol{\beta}} K_{ij}$$ [6]

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_{\boldsymbol{\beta}} \gamma_{ij}$$ [6]

$$R + K^2 - K_{ij}K^{ij} = 16\pi e$$ [1]

$$D_j K^j{}_i - D_i K = 8\pi j_i$$ [3]

These 6+6+4=**16** eqs. Einstein equations are **10** second-order equations in time or **20** first-order in time. Where are the missing four equations?…

NOTE: the lapse, and shift are not solutions of the Einstein equations but represent our "gauge freedom", namely the freedom (arbitrariness) in which we choose to foliate the spacetime.

Any prescribed choice for the lapse is usually referred to as a "slicing condition", while any choice for the shift is usually referred to as "spatial gauge condition"

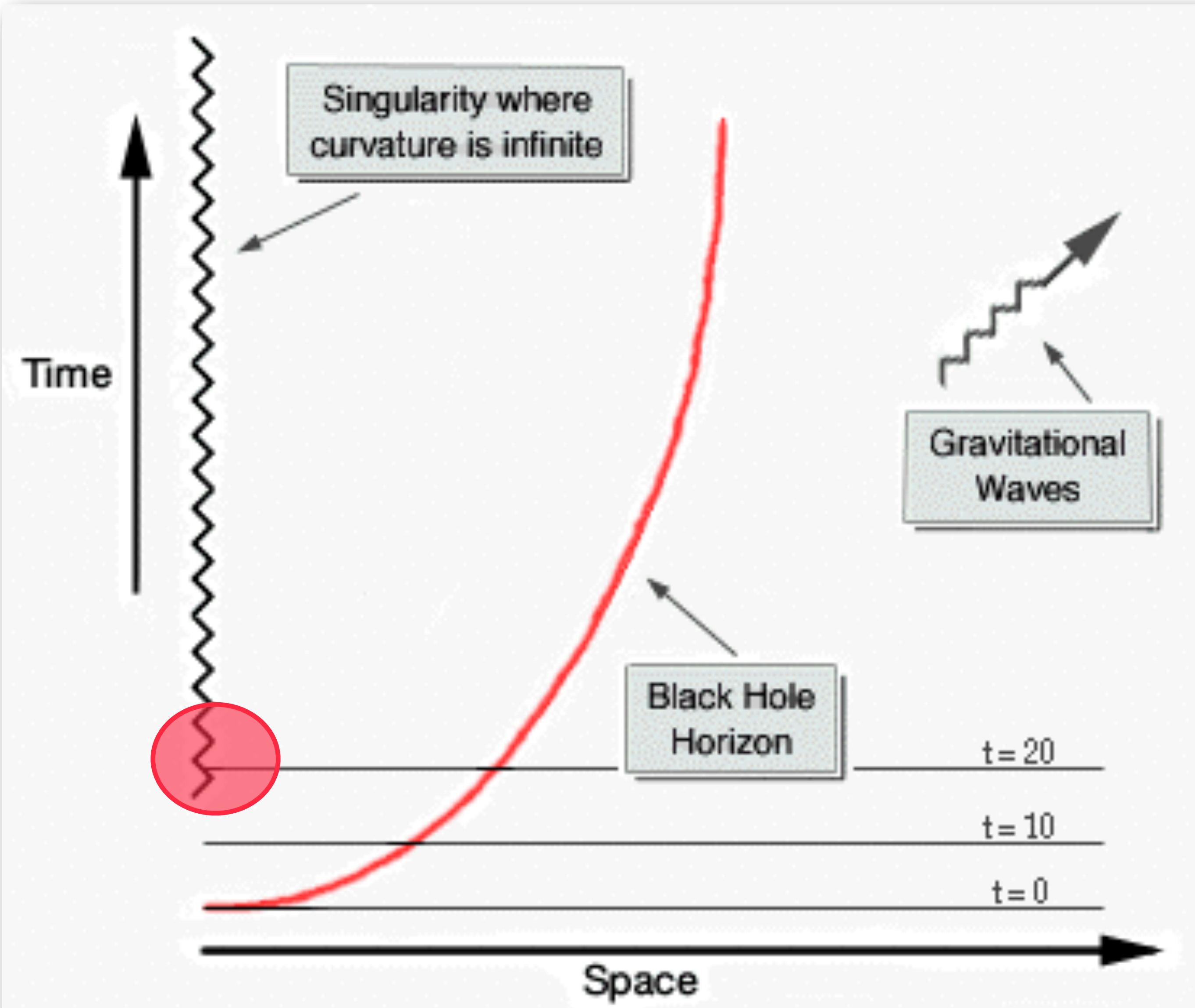
While there are infinite possible choices, not all of them are equally useful to carry out numerical simulations. Indeed, there is a whole branch of numerical relativity that is dedicated to finding suitable gauge conditions.

Several possible routes are possible

i) make a guess (i.e. prescribe a functional form) for the lapse, and shift and hope for the best: eg geodesic slicing $\alpha = 1\,,\ \beta^i = 0$

obviously not a good idea

Choosing the right temporal gauge



Suppose you want to follow the gravitational collapse to a bh and assume a simplistic gauge choice (geodesic slicing):

$$\alpha = 1, \quad \beta^i = 0$$

That would lead rapidly ($t \approx \pi M$) to a code crash No chance of measuring GWs which need to be collected on timescales $t \sim 10^3\, M$!

Several possible routes are possible

i) make a guess (i.e. prescribe a functional form) for the lapse, and shift and hope for the best: eg geodesic slicing $\alpha = 1\,,\;\; \beta^i = 0$

obviously not a good idea

ii) fix the lapse, and shift by requiring they satisfy some condition: eg maximal slicing for the lapse

$$\partial_t K = 0 \quad \Longrightarrow \quad D^i D_i \alpha = \alpha[K_{ij}K^{ij} + 4\pi(e+S)]$$

which has the desired "singularity-avoiding" properties. Similarly, the minimal distortion shift condition guarantees minimizes the changes in the conformally related metric

Good idea mathematically (the coordinates do exactly what they should). Unfortunately this leads to elliptic eqs which are computationally too expensive to solve at each time

iii) fix the lapse, and shift dynamically by requiring they satisfy comparatively simple evolution equations

This is the common solution. The advantage is the eqs for the lapse and shift are simple time evolution eqs.

A family of slicing conditions that works very well to obtain both a strongly hyperbolic evolution eqs. and for stable numerical evolutions is the Bona-Masso slicing

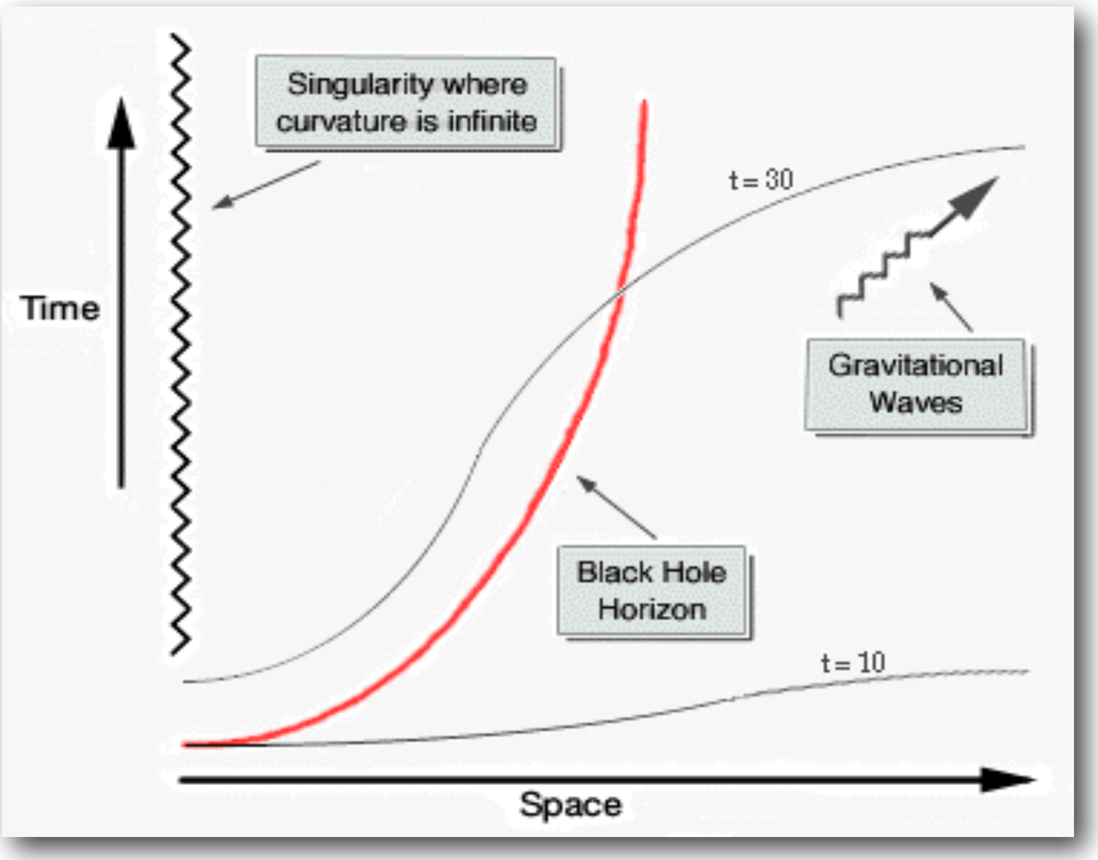
$$\partial_t \alpha - \beta^i \partial_i \alpha = -\alpha^2 f(\alpha)(K - K_0)$$

where $K_0 \equiv K(t=0)$ and $f(\alpha)$ is positive but otherwise arbitrary.

Indeed the condition $\partial_t \alpha \sim -\alpha^2 f(\alpha) K$ represents a family of slicing conditions such that:

$$
\begin{aligned}
f = 0\,, (\alpha = 1 \text{ initially})\,, \quad & : \text{geodesic slicing} \\
f = 1\,, \quad & : \text{harmonic slicing} \\
f = 2/\alpha\,, \quad & : \text{"}1+\log\text{" slicing} \\
f \to \infty\,, \quad & : \text{maximal slicing}
\end{aligned}
$$

The "1+log" slicing condition also has excellent singularity avoiding properties since $\partial_t \alpha \sim -\alpha$ and hence the lapse remains very small in those regions where it has "collapsed" to small values



Similarly, a popular choice for the shift is the hyperbolic "Gamma-driver" condition

$$\partial_t \beta^i - \beta^j \partial_j \beta^i = \frac{3}{4} \alpha B^i \,,$$
$$\partial_t B^i - \beta^j \partial_j B^i = \partial_t \tilde{\Gamma}^i - \beta^j \partial_j \tilde{\Gamma}^i - \eta B^i \,,$$

where η acts as a restoring force to avoid large oscillations in the shift and the driver tends to keep the Gammas constant (reminiscent of minimal distortion)

Overall, the "1+log" slicing condition and the "Gamma-driver" shift condition are the most widely used both in vacuum and non-vacuum spacetimes

Extraction of gravitational waves

Wave-extraction techniques

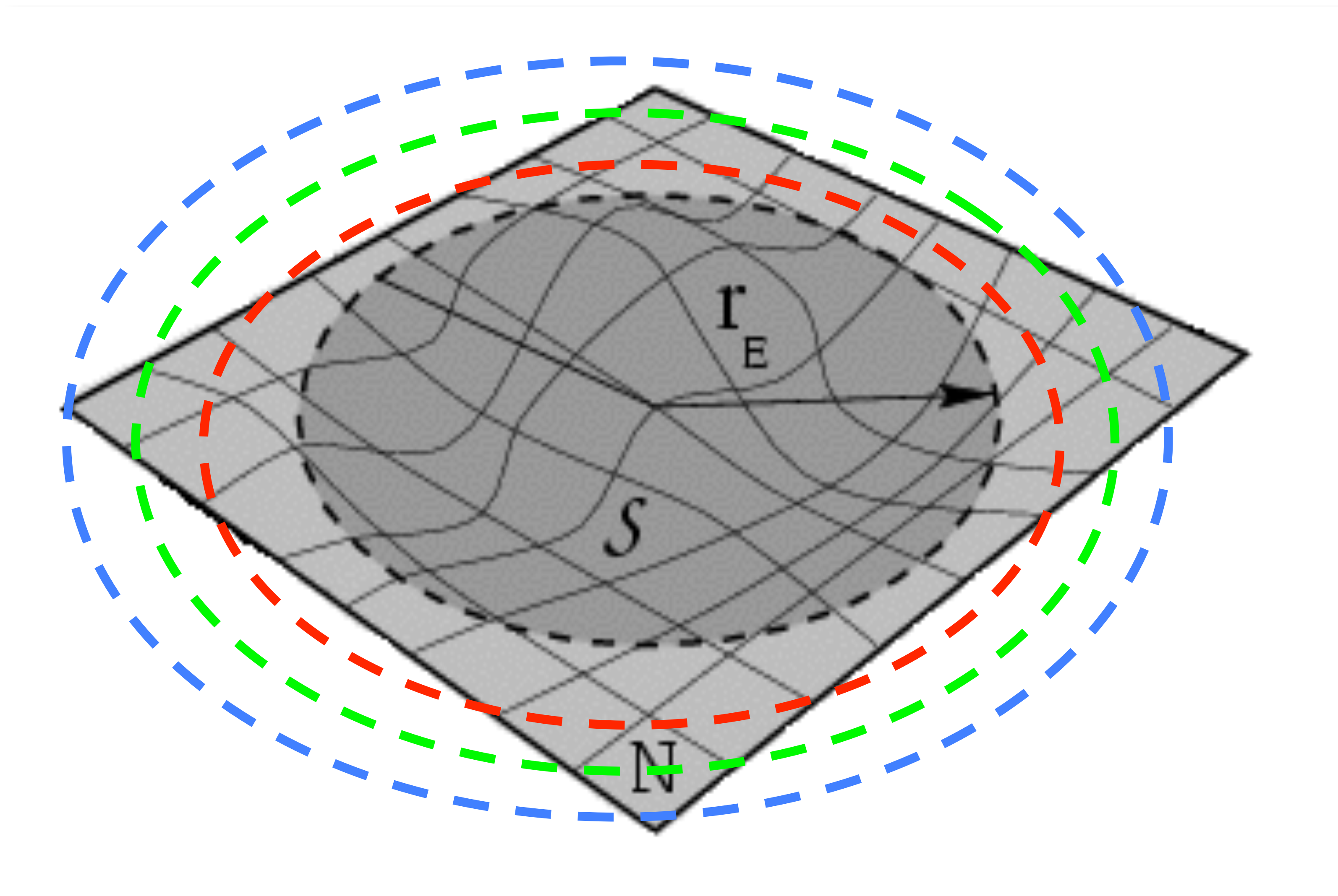
Computing the waveforms is the ultimate goal of most numerical relativity and there are several ways of extracting GWs from numerical relativity codes:

- asymptotic measurements
  - null slicing
  - conformal compactification
- non-asymptotic measurements (finite-size extraction worldtube)
  - Weyl scalars
  - perturbative matching to a Schwarzschild background

All have different degrees of success and this depends on the efficiency of the process which is very different for different sources $\frac{\Delta M}{M} \sim 10^{-2} \text{ (binary bhs)} - 10^{-7} \text{ (collapse to bh)}$

Wave-extraction techniques

In both approaches, "*observers*" are placed on nested 2-spheres and calculate there either the Weyl scalars or decompose the metric into tensor spherical-harmonics to calculate the gauge-invariant perturbations of a Schwarzschild black hole



Once the waveforms are calculated, all the related quantities: energy, momentum and angular momentum radiated can be derived simply.

Gauge-invariant pertubations

Similarly, at a sufficiently large distance from the source and assuming the spacetime resembles that of a Schwarzschild BH

$$h_+ - \mathrm{i}h_\times = \frac{1}{\sqrt{2}r}\sum_{l,m}\left(Q^+_{\ell m} - \mathrm{i}\int_{-\infty}^{t} Q^\times_{\ell m}(t')\mathrm{d}t'\right)Y^{-2}_{\ell m} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

where $Q^\times_{\ell\, m}$, $Q^+_{\ell m}$ are the odd and even-parity *gauge-invariant* perturbations of a Schwarschild bh. The projection of the momenutm flux on the equatorial plane is, for instance,

$$\mathcal{F}^{\ell m}_x + \mathrm{i}\mathcal{F}^{\ell m}_y \equiv \frac{(-1)^m}{16\pi\ell(\ell+1)}\Bigg\{-2\mathrm{i}\left[a_{\ell m}\dot{Q}^+_{\ell -m}Q^\times_{\ell\, m-1} + b_{\ell m}\dot{Q}^+_{\ell m}Q^\times_{\ell\, -(m+1)}\right]$$
$$+\sqrt{\frac{\ell^2(\ell-1)(\ell+3)}{(2\ell+1)(2\ell+3)}}\Bigg[c_{\ell m}\left(\dot{Q}^+_{\ell\, -m}\dot{Q}^+_{\ell+1\; m-1} + Q^\times_{\ell\, -m}\dot{Q}^\times_{\ell+1\; m-1}\right)$$
$$+d_{\ell m}\left(\dot{Q}^+_{\ell m}\dot{Q}^+_{\ell+1\; -(m+1)} + Q^\times_{\ell m}\dot{Q}^\times_{\ell+1\; -(m+1)}\right)\Bigg]\Bigg\},$$

This quantity can be used, for instance, to calculate the recoil.

Weyl scalars

The Newman-Penrose formalism provides a convenient representation for radiation-related quantities as spin-weighted scalars. In particular, the component of the Weyl tensor

$$\Psi_4 \equiv -C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta,$$

can be associated with the gravitational-radiation content of the spacetime because it falls like ~$1/r$. Here $(\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}})$, is a null frame and in practice we define an orthonormal polar coordinate basis $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\phi})$, centred on the Cartesian grid origin and with poles along $\boldsymbol{e}_z$.
Using then the normal to the slice as $\boldsymbol{e}_0 = \hat{\boldsymbol{t}}$ the components of the frame are

$$\boldsymbol{l} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{t}} - \hat{\boldsymbol{r}}), \quad \boldsymbol{n} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{t}} + \hat{\boldsymbol{r}}), \quad \boldsymbol{m} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\theta}} - \mathrm{i}\hat{\boldsymbol{\phi}})$$

Weyl scalars

We then calculate Ψ_4 in terms of ADM related quantities as

$$\Psi_4 = C_{ij}\bar{m}^i\bar{m}^j,$$

where

$$C_{ij} \equiv R_{ij} - KK_{ij} + K_i{}^k K_{kj} - \mathrm{i}\epsilon_i{}^{kl}\nabla_l K_{jk}.$$

Then at a sufficiently large distance from the source the GWs in the two polarizations $h_\times, h_+$ can be written as

$$h_+ - \mathrm{i}h_\times = \lim_{r\to\infty}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t''\Psi_4$$

Then, eg, the projection of the momentum flux on the equatorial plane as

$$\mathcal{F}_i = \frac{\mathrm{d}P_i}{\mathrm{d}t} = \lim_{r\to\infty}\left\{\frac{r^2}{16\pi}\int d\Omega\, n_i \left|\int_{-\infty}^t dt\Psi_4\right|^2\right\}.$$

This quantity can be used, for instance, to calculate the recoil.

Recap (I)

- ☑ The 3+1 splitting of the 4-dim spacetime represents an effective way to perform numerical solutions of the Einstein eqs.
- ☑ Such a splitting amounts to projecting all 4-dim. tensors either on spatial hypersurfaces or along directions orthogonal to such hypersurfaces.
- ☑ The 3-metric and the extrinsic curvature describe the properties of each slice.
- ☑ Two functions, the lapse and the shift, tell how to relate coordinates between two slices: the lapse measures the proper time, while the shift measures changes in the spatial coords.
- ☑ Einstein equations naturally split into evolution equations and constraint equations.

Recap (II)

- ☑ The ADM eqs are ill posed and not suitable for numerics.
- ☑ Alternative formulations (BSSNOK, CCZ4, Z4c) have been developed that are strongly hyperbolic and hence well-posed.
- ☑ Both formulations make use of the constraint equations and can use additional evolution equations to damp the violationsns
- ☑ The hyperbolic evolution eqs. to solve are: 6+6+(3+1+1) = 17. We also "compute" 3+1=4 elliptic constraint eqs

$$\mathcal{H} \equiv {}^{(3)}R + K^2 - K_{ij}K^{ij} = 0\,, \quad \text{(Hamiltonian constraint)}$$

$$\mathcal{M}^i \equiv D_j(K^{ij} - g^{ij}K) = 0\,, \quad \text{(momentum constraints)}$$

NOTE: these eqs are not solved but only monitored to verify

$$||\mathcal{H}|| \simeq ||\mathcal{M}^i|| < \varepsilon \sim 10^{-4} - 10^{-2}$$

Recap (III)

- ☑ The lapse and the shift have simple physical definitions and relate events on two different hypersurfaces.
- ☑ Getting a good formulation of the Einstein eqs will work only in conjunction with good gauge conditions. "1+log" slicing "Gamma-driver" conditions work well in a number of conditions.
- ☑ Even with suitable formulations and gauge conditions, any astrophysical prediction needs the calculation of "realistic" initial data and hence the solution of elliptic equations.
- ☑ GWs can be extracted with great accuracy. Several methods using either the radiative part of the Riemann tensor or perturbations of the Schwarzschild spacetime.