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First-order stable and causal hydrodynamics from kinetic theory

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Motivation

- *In relativistic heavy-ion collisions, a deconfined state of matter is produced, known as Quark-Gluon Plasma (QGP).*
- *The kinetic theory and hydrodynamics have been essential effective models to understand the evolution of this system.*
- *Relativistic generalization of Navier-Stokes equations known to be* ***acausal*** *and* ***unstable*** *- Higher order theories are introduced.*
- *A new solution to the causality and stability problems for* ***first-order theory*** *recently proposed by Bemfica, Disconzi, Noronha, and Kovtun (BDNK).*

Microscopic Theory

The relativistic Boltzmann transport equation:

$$p^\mu \partial_\mu f(x,p) = C[f] = \int d\Gamma_{p_1} d\Gamma_{p'} d\Gamma_{p'_1} \left[f_{p'} f_{p'_1} (1 \pm f_p)(1 \pm f_{p_1}) - f_p f_{p_1} (1 \pm f_{p'}) \left(1 \pm f_{p'_1}\right) \right] W(p' p'_1 | p p_1)$$

The distribution function for small deviation from local equilibrium:

$$f_p = f_p^{(0)} + \delta f_p, \quad \delta f_p = f_p^{(0)} (1 \pm f_p^{(0)}) \phi_p$$

The collision operator in linear order becomes:

$$C[f] = -\mathcal{L}[\phi] = \int d\Gamma_{p_1} d\Gamma_{p'} d\Gamma_{p'_1} f_p^{(0)} f_{p_1}^{(0)} (1 \pm f_{p'}^{(0)})(1 \pm f_{p'_1}^{(0)}) [\phi_p + \phi_{p_1} - \phi_{p'} - \phi_{p'_1}] W(p' p'_1 | p p_1)$$

Properties of the linearized collision operator:

Self-Adjoint

$$\int d\Gamma_p \psi \mathcal{L}[\phi] = \int d\Gamma_p \phi \mathcal{L}[\psi]$$

Microscopic Conservation

$$\mathcal{L}[a + b_\mu p^\mu] = 0$$

Summation Invariant

$$\int d\Gamma_p \mathcal{L}[\phi] = 0, \quad \int d\Gamma_p p^\mu \mathcal{L}[\phi] = 0$$

The out-of-equilibrium distribution function

In general, the out-of-equilibrium distribution function can be expressed as

S. R. de Groot, Relativistic Kinetic Theory

$$\phi^{(r)} = \sum_l A_l^{(r)} X^{(r)l} + \sum_m B_m^{(r)\mu} Y_\mu^{(r)m} + \sum_n C_n^{(r)\mu\nu} Z_{\mu\nu}^{(r)n}$$

- *Field gradients:* $X^{(r)l},\ Y_\mu^{(r)m},\ Z_{\mu\nu}^{(r)n}$
- *The unknown coefficients:* $A_l^{(r)},\ B_m^{(r)\mu},\ C_n^{(r)\mu\nu}$

The unknown coefficients are the function of space-time, particle momentum, rest mass and temperature. We expand the coefficients a polynomial basis as:

$$A_l^{(r)} = \sum_{s=0}^{\infty} A_l^{r,s}(z,x) P_s^{(0)} \qquad B_m^{(r)\mu} = \sum_{s=0}^{\infty} B_m^{r,s}(z,x) P_s^{(1)} \tilde{p}^{\langle\mu\rangle} \qquad C_n^{(r)\mu\nu} = \sum_{s=0}^{\infty} C_n^{r,s}(z,x) P_s^{(2)} \tilde{p}^{\langle\mu}\tilde{p}^{\nu\rangle}$$

Hydrodynamic Frame

General hydrodynamic frame or general matching conditions:

$$\int dF_{\rm p}\tilde{E}^i_{\rm p}\phi = 0, \quad \int dF_{\rm p}\tilde{E}^j_{\rm p}\phi = 0, \quad \int dF_{\rm p}\tilde{E}^k_{\rm p}\tilde{p}^{\langle\mu\rangle}\phi = 0$$

For Landau-Lifshitz (LL) frame ***(i,j,k) = (2,1,1)*** *and for Eckart frame* ***(i,j,k) = (2,1,0)***.

For the properties of linearized collision operator, we can't determine homogeneous solutions from transport equation. Using ***general matching*** *conditions we calculate the* ***homogeneous part***:

$$A_l^{r,0} = -\frac{1}{\mathcal{D}^{0,1}_{i,j}}\sum_{s=2}^{\infty}\mathcal{D}^{s,1}_{i,j}A_l^{r,s}, \quad A_l^{r,1} = -\frac{1}{\mathcal{D}^{1,0}_{i,j}}\sum_{s=2}^{\infty}\mathcal{D}^{s,0}_{i,j}A_l^{r,s}, \quad B_m^{r,0} = -\frac{1}{J_k}\sum_{s=1}^{\infty}J_{k+s}B_m^{r,s}$$

The rests are known as ***inhomogeneous/interaction*** *solution can be estimated from the transport equation.*

$$\mathcal{D}^{m,n}_{i,j} = I_{i+m}I_{j+n} - I_{i+n}I_{j+m} \qquad I_n = \int dF_p\tilde{E}^n_{\rm p}, \Delta^{\mu\nu}J_n = \int dF_p\tilde{p}^{\langle\mu\rangle}\tilde{p}^{\langle\nu\rangle}\tilde{E}^n_{\rm p}$$

Homogeneous and Inhomogeneous Solution

*The entire out-of-equilibrium distribution function for any order can be determined separately for **frame (homogeneous)** and **interaction (inhomogeneous)** parts as:*

$$\phi^{(r)} = \phi^{(r)}_{\rm int} + \phi^{(r)}_{\rm h}$$

The homogeneous part:

$$\phi^{(r)}_{\rm h} = -\tilde{E}_{\rm p}\left[\frac{I_j}{\mathcal{D}^{1,0}_{i,j}}\int dF_p \tilde{E}^i_{\rm p}\phi^{(r)}_{\rm int} + (i \leftrightarrow j)\right] - \left[\frac{I_{j+1}}{\mathcal{D}^{0,1}_{i,j}}\int dF_p \tilde{E}^i_{\rm p}\phi^{(r)}_{\rm int} + (i \leftrightarrow j)\right] - \frac{\tilde{p}_{\langle\nu\rangle}}{J_k}\int dF_p \tilde{E}^k_{\rm p}\tilde{p}^{\langle\nu\rangle}\phi^{(r)}_{\rm int}$$

The inhomogeneous/interaction part:

$$\phi^{(r)}_{\rm int} = \sum_l \sum_{s=2}^{\infty} A^{r,s}_l P^{(0)}_s X^{(r)l} + \sum_m \sum_{s=1}^{\infty} B^{r,s}_m P^{(1)}_s \tilde{p}^{\langle\mu\rangle} Y^{(r)m}_{\mu} + \sum_n \sum_{s=0}^{\infty} C^{r,s}_n P^{(2)}_s \tilde{p}^{\langle\mu}\tilde{p}^{\mu\rangle} Z^{(r)n}_{\mu\nu}$$

Tensor decomposition of energy-momentum and particle current

General expressions for energy-momentum tensor and particle four-current:

$$T^{\mu\nu} = (\epsilon_0 + \delta\epsilon)\, u^\mu u^\nu - (P_0 + \delta P)\, \Delta^{\mu\nu} + (W^\mu u^\nu + W^\nu u^\mu) + \pi^{\mu\nu} \qquad N^\mu = (n_0 + \delta n)\, u^\mu + V^\mu$$

In kinetic theory, the energy-momentum tensor and particle four-current defined as:

$$T^{\mu\nu} = \int d\Gamma_{\mathrm{p}} p^\mu p^\nu f(x,p) \qquad N^\mu = \int d\Gamma_{\mathrm{p}} p^\mu f(x,p)$$

From the general tensor decomposition it follows:

$$\epsilon_0 = \int d\Gamma_{\boldsymbol{p}} E_{\mathrm{p}}^2 f_{\boldsymbol{p}}^{(0)} \qquad n_0 = \int d\Gamma_{\boldsymbol{p}} E_{\mathrm{p}} f_{\boldsymbol{p}}^{(0)}$$

$$W^\mu = \int d\Gamma_p E_{\mathrm{p}} p^{\langle\mu\rangle} \delta f_p \qquad V^\mu = \int d\Gamma_p p^{\langle\mu\rangle} \delta f_p$$

System Dynamics

We use the ***momentum-dependent*** *relaxation time approximation (MDRTA) for solving the relativistic transport equation as a model study.*

K. Dusling et al., PRC 81, 034907 (2010)
G. S. Rocha et al., PRL 127, 042301 (2021)
S. Mitra, PRC 103, 014905 (2021)
D. Dash et al., PLB 831, 137202 (2022)

$$\tilde{p}^{\mu}\partial_{\mu}f = -\frac{\tilde{E}_{\rm p}}{\tau_R} f^{(0)}(1 \pm f^{(0)})\phi_{\rm int} \qquad \tau_R(x,p) = \tau_R^0(x)\tilde{E}_{\rm p}^{\Lambda}$$

Appropriate collision kernel:

$$\mathcal{L}_{\rm MDRTA}[\phi] = \frac{(p\cdot u)}{\tau_R} f^{(0)}(1\pm f^{(0)})\left[\phi - \frac{\left\langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p^2\right\rangle\left\langle\frac{\tilde{E}_p}{\tau_R}\phi\right\rangle - \left\langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p\right\rangle\left\langle\frac{\tilde{E}_p}{\tau_R}\phi\tilde{E}_p\right\rangle}{\left\langle\frac{\tilde{E}_p}{\tau_R}\right\rangle\left\langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p^2\right\rangle - \left\langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p\right\rangle^2} - \tilde{E}_p\frac{\left\langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p\right\rangle\left\langle\frac{\tilde{E}_p}{\tau_R}\phi\right\rangle - \left\langle\frac{\tilde{E}_p}{\tau_R}\right\rangle\left\langle\frac{\tilde{E}_p}{\tau_R}\phi\tilde{E}_p\right\rangle}{\left\langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p\right\rangle^2 - \left\langle\frac{\tilde{E}_p}{\tau_R}\right\rangle\left\langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p^2\right\rangle} - \tilde{p}_{\langle\nu\rangle}\frac{\left\langle\frac{\tilde{E}_p}{\tau_R}\phi\tilde{p}^{\langle\nu\rangle}\right\rangle}{\frac{1}{3}\left\langle\frac{\tilde{E}_p}{\tau_R}\tilde{p}^{\langle\mu\rangle}\tilde{p}_{\langle\mu\rangle}\right\rangle}\right]$$

The inhomogeneous/interaction part:

Rocha et al., arXiv: 2205.00078

$$\phi_{\rm int}^{(1)} = -\tau_R^0\tilde{E}_{\rm p}^{\Lambda-1}\left[\tilde{E}_{\rm p}^2\frac{DT}{T} + \tilde{E}_{\rm p}D\tilde{\mu} + \left(\frac{\tilde{E}_{\rm p}^2}{3} - \frac{z^2}{3}\right)(\partial\cdot u) + \tilde{E}_{\rm p}\tilde{p}^{\langle\mu\rangle}\left(\frac{\nabla_{\mu}T}{T} - Du_{\mu}\right) + \tilde{p}^{\langle\mu\rangle}\nabla_{\mu}\tilde{\mu} - \tilde{p}^{\langle\mu}\tilde{p}^{\nu\rangle}\sigma_{\mu\nu}\right]$$

First-Order Field Redefinition

The first-order thermodynamic field corrections in a general frame written as:

F. S. Bemfica et al., PRD 98, 104064 (2018)
F. S. Bemfica et al., PRD 100, 104020 (2019)
P. Kovtun, JHEP 10, 034 (2019)

$$\delta\epsilon^{(1)} = \varepsilon_1 \frac{DT}{T} + \varepsilon_2 (\partial \cdot u) + \varepsilon_3 D\tilde{\mu}$$

$$\delta P^{(1)} = \pi_1 \frac{DT}{T} + \pi_2 (\partial \cdot u) + \pi_3 D\tilde{\mu}$$

$$\delta n^{(1)} = \nu_1 \frac{DT}{T} + \nu_2 (\partial \cdot u) + \nu_3 D\tilde{\mu}$$

$$V^{(1)\mu} = \gamma_1 \left(\frac{\nabla^\mu T}{T} - Du^\mu \right) + \gamma_3 \nabla^\mu \tilde{\mu}$$

$$W^{(1)\mu} = \theta_1 \left(\frac{\nabla^\mu T}{T} - Du^\mu \right) + \theta_3 \nabla^\mu \tilde{\mu}$$

Not ***all transport coefficients are invariant*** *under the general first-order field redefinition, the invariant ones are:*

$$f_i = \pi_i - \varepsilon_i \left(\frac{\partial P_0}{\partial \epsilon_0} \right)_{n_0} - \nu_i \left(\frac{\partial P_0}{\partial n_0} \right)_{\epsilon_0}$$

$$l_i = \gamma_i - \frac{n_0}{\epsilon_0 + P_0} \theta_i$$

$$\zeta = -f_2 + \left(\frac{\partial P_0}{\partial \epsilon_0} \right)_{n_0} f_1 + \frac{1}{T} \left(\frac{\partial P_0}{\partial n_0} \right)_{\epsilon_0} f_3$$

$$k_n = l_3 - \frac{n_0 T}{\epsilon_0 + P_0} l_1$$

$$\eta$$

The detailed expressions of bulk and shear viscosity and heat conductivity are given in

S. Mitra, PRC 105, 014902 (2022)

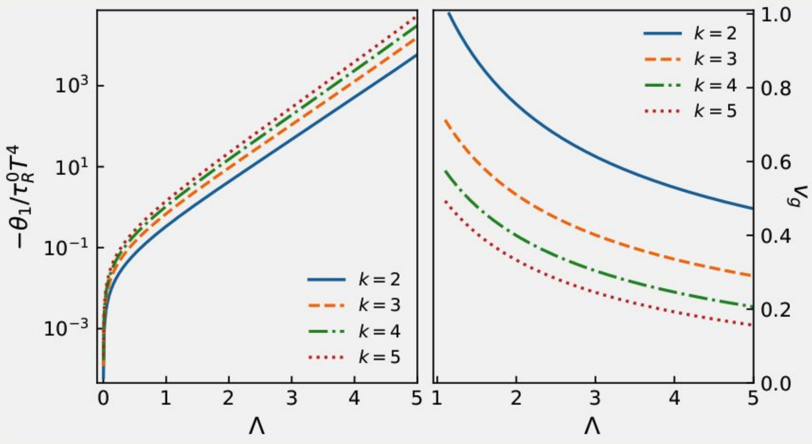
Shear Channel

*The **stability criteria** is:*

$$\theta_1 = -\tau_R^0 T^2 \left(J_{\Lambda+1} + \frac{\epsilon_0 + P_0}{T^2} \frac{J_{k+\Lambda}}{J_k} \right) < 0$$

*The **causality criteria** is:*

$$v_g = \sqrt{\frac{\eta}{\theta}} < 1 \qquad \textit{Here} \qquad \theta = -\theta_1$$



For certain range of microscopic interactions, the shear channel become acausal.

Sound Channel

Using ***Routh-Hurwitz criteria****, we find the following conditions for* ***stability*** *of the sound channel are:*

$$A_6 > 0\ ,\ A_5 > 0\ ,\ A_3^0 > 0\ ,\ B_2 = (A_4^0 A_5 - A_3^0 A_6)/A_5 > 0$$

All A_i *are function of the* ***transport coefficients****. Here we show few transport coefficients:*

$$\varepsilon_1 = \tau_R^0 \left[\frac{\partial \epsilon_0}{\partial \tilde{\mu}} \frac{\mathcal{D}_{i,j}^{\Lambda+1,1}}{\mathcal{D}_{i,j}^{0,1}} + T \frac{\partial \epsilon_0}{\partial T} \frac{\mathcal{D}_{i,j}^{\Lambda+1,0}}{\mathcal{D}_{i,j}^{1,0}} - T^2 I_{\Lambda+3} \right]$$

$$\varepsilon_3 = \tau_R^0 \left[\frac{\partial \epsilon_0}{\partial \tilde{\mu}} \frac{\mathcal{D}_{i,j}^{\Lambda,1}}{\mathcal{D}_{i,j}^{0,1}} + T \frac{\partial \epsilon_0}{\partial T} \frac{\mathcal{D}_{i,j}^{\Lambda,0}}{\mathcal{D}_{i,j}^{1,0}} - T^2 I_{\Lambda+2} \right]$$

$$\nu_1 = \tau_R^0 \left[\frac{\partial n_0}{\partial \tilde{\mu}} \frac{\mathcal{D}_{i,j}^{\Lambda+1,1}}{\mathcal{D}_{i,j}^{0,1}} + T \frac{\partial n_0}{\partial T} \frac{\mathcal{D}_{i,j}^{\Lambda+1,0}}{\mathcal{D}_{i,j}^{1,0}} - T I_{\Lambda+2} \right]$$

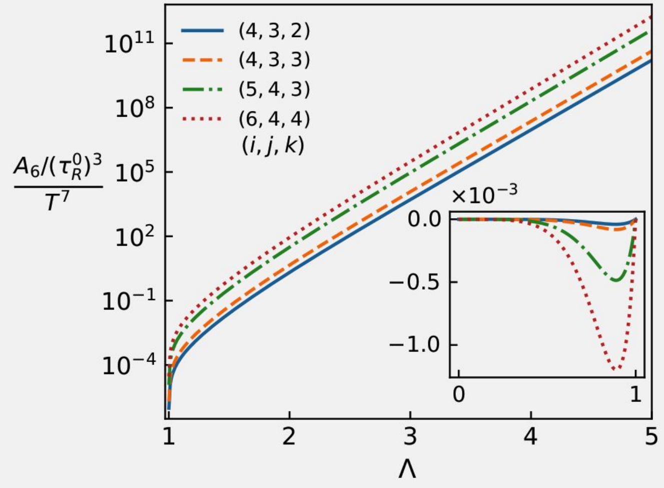
$$\nu_3 = \tau_R^0 \left[\frac{\partial n_0}{\partial \tilde{\mu}} \frac{\mathcal{D}_{i,j}^{\Lambda,1}}{\mathcal{D}_{i,j}^{0,1}} + T \frac{\partial n_0}{\partial T} \frac{\mathcal{D}_{i,j}^{\Lambda,0}}{\mathcal{D}_{i,j}^{1,0}} - T I_{\Lambda+1} \right]$$

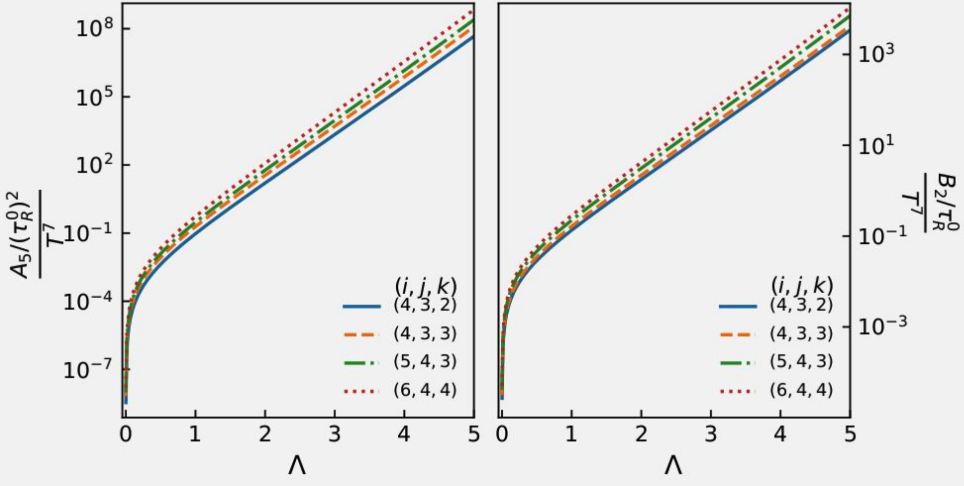
- ε_1, ν_1 *vanish both for i=1, j=2 (LL+Eckart) for all* Λ *, and also at* $\Lambda = 0$ *for all frame.*

- ε_3, ν_3 *obey the same but also vanish for* $\Lambda = 1$ *at all frames.*

Sound Channel

The coefficients make A_5 and A_4^0 vanish for $\Lambda = 0$ and A_6 vanish for both $\Lambda = 0$ and 1 at any frame.





Also the stability of sound modes becomes unstable for certain range of microscopic interactions.

Conclusion

- *A first-order, relativistic stable and causal hydrodynamic theory has been derived in a general frame from the Boltzmann transport equation*
- *We have shown that in order to hold stability and causality at first-order theories, besides a general frame, the system interactions need to be carefully taken into account.*
- *The conventional momentum independent RTA leads to acausality in general frame.*

Thank You!!