Thermalisation and Attractors in Ultrarelativistic Heavy-Ion Collisions

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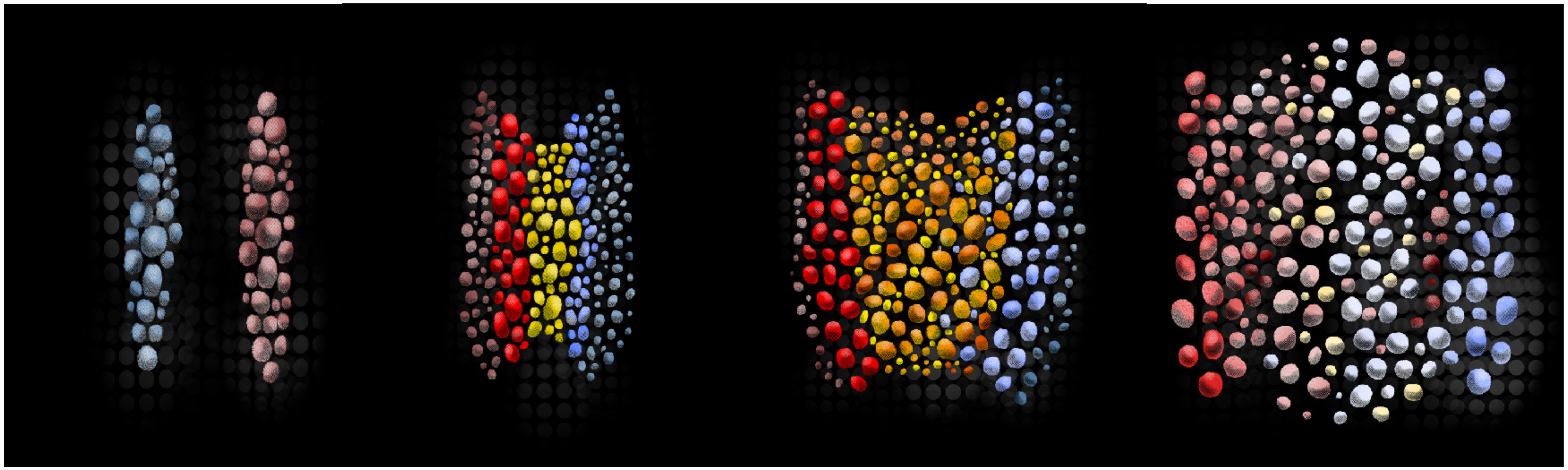
Abstract

Studies of quark-gluon plasma dynamics have led to a puzzle concerning the successful application of fluid dynamics to systems far from local equilibrium. I will describe a circle of ideas which has the potential to resolve this puzzle; it involves so called early-time attractors, which have been identified in a number of models of equilibration. I will also show how modern asymptotic methods have shed light on the emergence of hydrodynamic behaviour. Despite much activity in this area, there are many unanswered questions which I will try to highlight.

Ultra-relativistic heavy-ion collisions



Far from equilibrium Hydrodynamics Almost thermal

Initial state: far from equilibrium

Final state: almost thermal

Fluid dynamics is successful in the description of heavy ion collisions starting at times < 1 fm, where the system is still very anisotropic.

The Early Thermalisation Puzzle

In general, fluid dynamics requires local equilibrium.

Can local equilibrium be attained at very stages of QGP evolution?

Some options:

- Hydrodynamics is valid more generally than we thought
- QGP thermalises really fast – strong coupling, small MFP
- There is something special about the initial state of QGP
- A combination of the above

Hydrodynamic Attractors

An attractor: a dynamically distinguished submanifold of the space of solutions which is approached by generic solutions.

There is a hydrodynamic attractor at late times, close to equilibrium: generic flows will decay to solutions of the perfect fluid theory.

Why is there a hydrodynamic attractor already at very early times?

We will consider some toy models where this idea can be explored numerically as well as analytically.

We will first focus on models where the attractor is universal: a single solution which is rapidly approached by generic solutions.

Model Frameworks

for studies of thermalisation

- Kinetic theory:
  - weakly coupled quasiparticles
  - direct connection to hydrodynamics
- AdS/CFT (holography):
  - strongly coupled
  - perturbations of equilibrium simple to describe
- Fluid Dynamics:
  - based on conservation laws (spacetime symmetries)
  - applies generally regardless of coupling
  - captures universal near-equilibrium asymptotics

Relativistic hydrodynamics
for perfect fluids

The energy-momentum tensor for any ideal fluid:

$$\langle \hat{T}^{\mu\nu} \rangle \equiv T^{\mu\nu} = (\mathscr{E} + \mathscr{P}) u^\mu u^\nu + \mathscr{P} \eta^{\mu\nu}$$

for which we have the conservation law $\boxed{\nabla_\alpha T^{\alpha\beta} = 0}$

Four equations, four independent variables:

- local energy density $\mathscr{E}$
- fluid 4-velocity $(u^\mu) = (\gamma, \gamma \vec{v}), \quad \gamma \equiv 1/\sqrt{1 - v^2}$

Microscopic information enters through the equation of state

$$\mathscr{P} = \mathscr{P}(\mathscr{E}) \approx \mathscr{E}/3 \qquad \text{(by conformal symmetry)}$$

At sufficiently high energies we expect conformal invariance.

Conformal symmetry
and Weyl transformations

Conformal transformations: changes of coordinates which preserve the metric up to a coordinate-dependent factor.

Weyl transformations and Weyl weights:

$$g_{\mu\nu} \longrightarrow e^{-2\phi(x)} g_{\mu\nu} \qquad \Longleftrightarrow \qquad w(g_{\mu\nu}) = -2$$

Conformal theories are invariant under Weyl transformations

The energy momentum tensor is defined by

$$\delta S = -\frac{1}{2}\int d^4x \sqrt{|g|}\,\delta g_{\mu\nu} T^{\mu\nu}$$

In a conformal theory it is traceless:

$$\boxed{T^{\mu}_{\mu} = 0}$$

Conformal Hydrodynamics
of perfect fluids

$$T^{\mu\nu} = (\mathscr{E} + \mathscr{P})u^\mu u^\nu + \mathscr{P}\eta^{\mu\nu} = \frac{1}{3}\mathscr{E}(\eta^{\mu\nu} + 4u^\mu u^\nu)$$

Weyl weights:

$$w(\eta_{\mu\nu}) = -2, \quad w(\eta^{\mu\nu}) = 2, \quad w(u^\mu) = 1, \quad w(\mathscr{E}) = 4$$

Weyl-covariant derivatives:

$$\mathscr{D}_\mu \Psi \equiv \nabla_\mu \Psi + w(\Psi)\mathscr{A}_\mu \quad \longrightarrow \quad e^{-w(\Psi)\phi}\, \mathscr{D}_\mu \Psi$$

Connection

$$\mathscr{A}_\mu = u^\nu \nabla_\nu u_\mu + \frac{1}{3} u^\mu \nabla_\nu u^\nu, \qquad \mathscr{A}_\mu \longrightarrow \mathscr{A}_\mu - \nabla_\mu \phi$$

Conservation laws are Weyl-covariant

$$\nabla_\alpha T^{\alpha\beta} = 0 \quad \Longleftrightarrow \quad \mathscr{D}_\alpha T^{\alpha\beta} = 0$$

Relativistic hydrodynamics
incorporating dissipation

To account for entropy production, we include dissipative terms:

$$T^{\mu\nu} = \frac{1}{3}\mathscr{E}(\eta^{\mu\nu} + 4u^\mu u^\nu) \boxed{+ \Pi^{\mu\nu}}$$

We can parameterise the dissipative tensor as

$$\Pi^{\mu\nu} = \pi^{\mu\nu} + \frac{1}{3}\mathscr{B}\left(\eta^{\mu\nu} + 4u^\mu u^\nu\right) + \mathscr{Q}^\mu u^\nu + \mathscr{Q}^\nu u^\mu$$

$$\pi_{\mu\nu}u^\nu = 0, \quad \mathscr{Q}_\mu u^\mu = 0$$

We can choose to impose the Landau frame condition

$$\mathscr{B} = 0, \quad \mathscr{Q}^\mu = 0$$

Relativistic hydrodynamics
incorporating dissipation

In the Landau frame we have

$$T^{\mu\nu} = \frac{1}{3}\mathcal{E}(\eta^{\mu\nu} + 4u^\mu u^\nu) + \pi^{\mu\nu}$$

Then the unique term with just one derivative is

$$\pi^{\mu\nu} = -\,\eta\,\sigma^{\mu\nu} \equiv -\,\eta\,\left(\mathscr{D}^\mu u^\nu + \mathscr{D}^\nu u^\mu\right)$$

where η is the shear viscosity, a function of the effective temperature defined through the equation

$$\mathcal{E} = \alpha T^4$$

This is the relativistic Navier-Stokes theory in the Landau frame.

Relativistic hydrodynamics
incorporating dissipation

Landau frame NS theory is acausal, and global equilibrium is unstable.

Two ways to recover stability and causality:

- Mueller-Israel-Stewart (MIS)
- Bemfica-Disconzi-Noronha-Kovtun (BDNK)

Here we will discuss the first option, which treats the dissipative tensor as an independent variable with an evolution equation

$$\tau_\pi \mathscr{D} \pi^{\mu\nu} + \pi^{\mu\nu} = -\eta \sigma^{\mu\nu}$$

where $\mathscr{D} \equiv u^\mu \mathscr{D}_\mu$.

This introduces a relaxation time $\tau_\pi(T) \equiv C_\tau / T$.

Relativistic hydrodynamics
linear perturbations of equilibrium

Linearised EOM: $L\delta\Phi = 0, \quad \delta\Phi \sim e^{-i\omega t + ikz}$

Both hydrodynamic and nonhydrodynamic modes appear, e.g.

$$\omega_{\mathrm{H}}^{(\pm)} = \pm \frac{k}{\sqrt{3}} - \frac{2i}{3\,T}\frac{\eta}{s}k^2 + \ldots \qquad \omega_{\mathrm{NH}} = -\,i\left(\frac{1}{\tau_\pi} - \frac{4}{3\,T}\frac{\eta}{s}k^2\right) + \ldots$$

Stability of equilibrium requires $\eta > 0, \quad \tau_\pi > 0$

Causality at the linearised level requires:

$$v = \frac{1}{\sqrt{3}}\sqrt{1 + 4\frac{\eta/s}{T\tau_\pi}} < 1 \qquad \Longleftrightarrow \qquad T\tau_\pi > 2\eta/s$$

The nonhydrodynamic mode is a (non-universal) regulator.

As the regulator is removed, the acausal NS theory is recovered.

Relativistic hydrodynamics
and the gradient expansion

The MIS relaxation equation

$$\tau_\pi \mathscr{D} \pi^{\mu\nu} + \pi^{\mu\nu} = -\,2\eta\sigma^{\mu\nu}$$

can be formally solved as an infinite series graded by the number of spacetime derivatives of the hydrodynamic variables:

$$\pi^{\mu\nu} = -\,\eta\sigma^{\mu\nu} + \tau_\pi \mathscr{D}\left(\eta\sigma^{\mu\nu}\right) + \ldots$$

Truncation of this series at order one recovers Navier-Stokes theory.

This is a special case of the hydrodynamic gradient expansion, which contains all possible terms allowed by symmetries with arbitrary transport coefficients, which parameterise the possible asymptotics.

Models such as MIS generate a subset of possible terms.

Relativistic hydrodynamics
and matching

In analogy with effective field theory ideas, we view hydrodynamics as an *effective description* to be matched with an underlying microscopic theory.

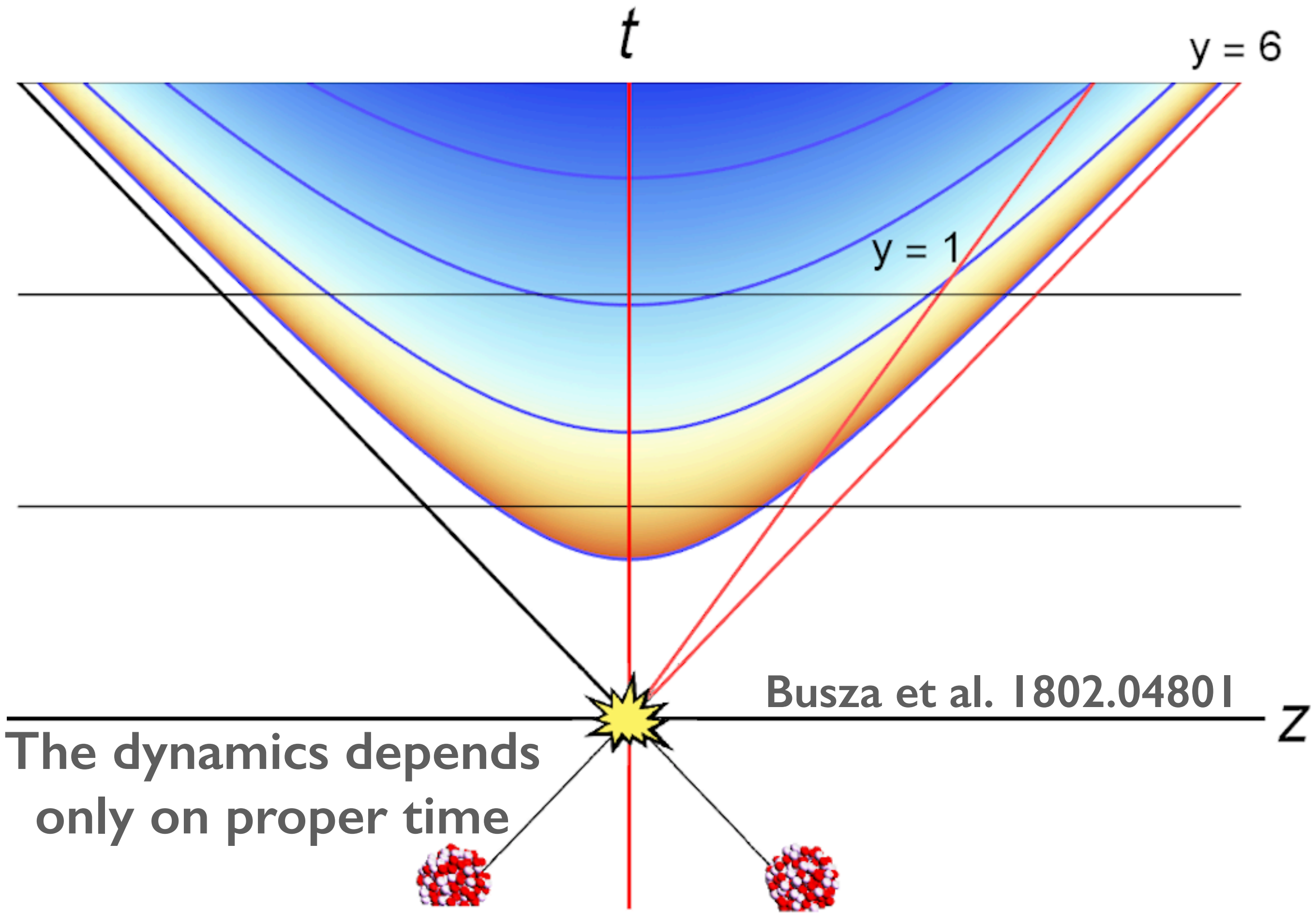
For example, an AdS/CFT calculation of the gradient expansion up to 2nd order in N=4 supersymmetric Yang-Mills theory can be reproduced by the MIS model, provided that the relaxation equation is supplemented by additional terms such that it generates all second order terms with independent coefficients. One then finds

$$\frac{\eta}{s} = \frac{1}{4\pi}, \qquad C_\tau = \frac{2 - \log 2}{2\pi}$$

with analogous results for some additional transport coefficients.

Bjorken flow

the spacetime picture



- Ultra-relativistic limit

$$\gamma \gg 1$$

- Pointlike collision
- Rapidity independence
- Milne coordinates

$$\boxed{t = \tau \cosh y, \quad z = \tau \sinh y}$$

- Metric $$ds^2 = -d\tau^2 + \tau^2 dy^2 + d\vec{x}_\perp^{\,2}$$
- Lab coordinates $$(u^\mu) = \gamma\,(1,0,0,z/t) = (t/\tau,0,0,z/\tau)$$
- Milne coordinates $$(u^\mu) = (1,0,0,0)$$

Bjorken flow
in MIS theory

The energy-momentum tensor

$$(T^\mu_\nu) = \underbrace{\mathrm{diag}(-\mathscr{E}, \mathscr{P}, \mathscr{P}, \mathscr{P})}_{\text{Perfect fluid}} + \underbrace{\mathrm{diag}(0, -\phi, \phi/2, \phi/2)}_{(\pi^\mu_\nu)}$$

It is convenient to introduce

$$(T^\mu_\nu) \equiv \mathrm{diag}(-\mathscr{E}, \mathscr{P}_L, \mathscr{P}_T, \mathscr{P}_T)$$

$$\mathscr{P}_L \equiv \frac{\mathscr{E}(\tau)}{3}\left(1 - \frac{2}{3}\mathscr{A}(\tau)\right), \qquad \mathscr{P}_T \equiv \frac{\mathscr{E}(\tau)}{3}\left(1 + \frac{1}{3}\mathscr{A}(\tau)\right)$$

The pressure anisotropy $\boxed{\mathscr{A} \equiv \frac{\mathscr{P}_T - \mathscr{P}_L}{\mathscr{P}} = \frac{9}{2}\frac{\phi}{\mathscr{E}}}$

Bjorken flow
in MIS theory

The evolution equations are:

$$\underbrace{\tau\partial_\tau\mathcal{E} + \frac{4}{3}\mathcal{E} = \phi,}_{\text{Conservation}} \qquad \underbrace{\tau_\pi\left(\partial_\tau + \frac{4}{3\tau}\right)\phi + \phi = \frac{4}{3\tau}\eta}_{\text{Relaxation}}$$

In a conformal theory

$$\eta = C_\eta s, \qquad \mathcal{E} = \alpha T^4, \qquad \tau_\pi = C_\tau / T$$

and we also use the thermodynamic relation

$$Ts = \frac{4}{3}\mathcal{E},$$

It will be useful to first eliminate $\phi(\tau)$ using the conservation equation to have an equation for the temperature alone.

Bjorken flow
in MIS theory

The temperature satisfies a single second order ODE:

$$C_\tau \left(\tau \left(\frac{\ddot{T}}{T} + 3 \left(\frac{\dot{T}}{T} \right)^2 \right) + \frac{11}{3} \frac{\dot{T}}{T} + \frac{4}{9\tau} \right) - \frac{4C_\eta}{9\tau} + \underbrace{\tau\dot{T} + \frac{1}{3}T}_{\text{perfect fluid}} = 0$$

Solutions have the asymptotic late proper time expansion

$$T(\tau) = \frac{\Lambda}{(\Lambda\tau)^{1/3}} \left(1 - \frac{2C_\eta}{3} \frac{1}{(\Lambda\tau)^{2/3}} - \frac{2C_\eta C_\tau}{9} \frac{1}{(\Lambda\tau)^{4/3}} + \ldots \right)$$

where Λ is an integration constant which contains the part of the information about the initial state which survives until late times.

Half of the information is not visible in this asymptotic limit.

Bjorken flow
in MIS theory

It will now be convenient to switch to dimensionless variables

$$(w, \mathscr{A}) \equiv (\tau T, \frac{9}{2}\frac{\phi}{\mathscr{E}})$$

The pressure anisotropy "measures distance from equilibrium".

Note that the asymptotic behaviour of the temperature implies

$$w(\tau) = (\Lambda\tau)^{2/3}\left(1 - \frac{2C_\eta}{3}\frac{1}{(\Lambda\tau)^{2/3}} - \frac{2C_\eta C_\tau}{9}\frac{1}{(\Lambda\tau)^{4/3}} + \dots\right)$$

so that one may view this variable as a dimensionless replacement for the proper time, at least for late times.

One can view it as the proper time in units of the relaxation time:

$$\tau/\tau_\pi = w/C_\tau$$

Bjorken flow
in MIS theory

The evolution equations expressed in the dimensionless variables:

$$\underbrace{\tau\,\partial_\tau \ln w = \frac{2}{3} + \frac{1}{18}\mathscr{A},}_{\text{Conservation}} \qquad \underbrace{C_\tau \tau\,\partial_\tau \mathscr{A} = 8C_\eta - w\mathscr{A} - \frac{2}{9}\mathscr{A}^2}_{\text{Relaxation}}$$

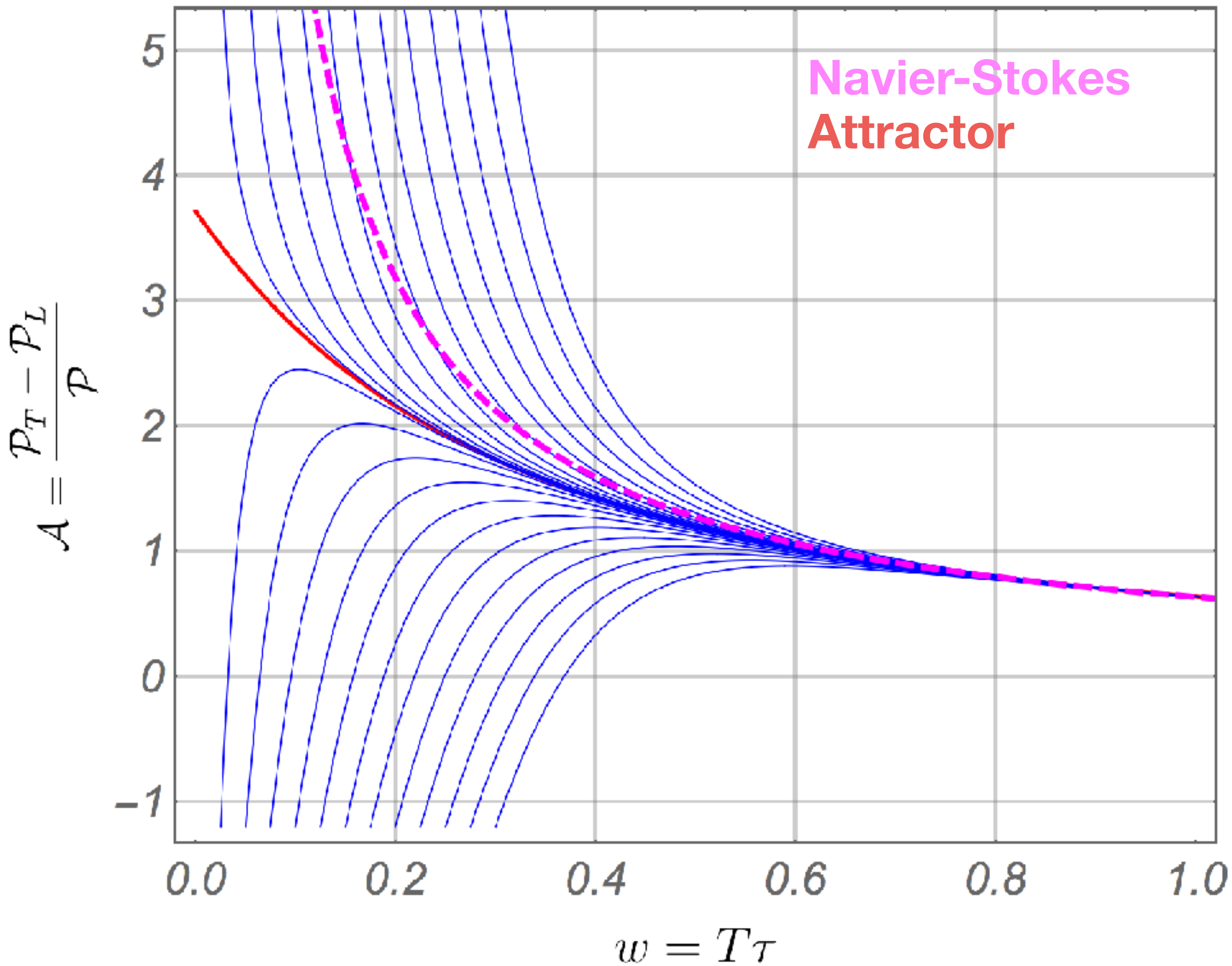
The phase portrait is determined by

$$C_\tau\left(1 + \frac{\mathscr{A}}{12}\right)\mathscr{A}' + \frac{C_\tau}{3w}\mathscr{A}^2 = \frac{3}{2}\left(\frac{8C_\eta}{w} - \mathscr{A}\right)$$

The conservation equation also implies

$$\frac{d\log T}{d\log w} = \frac{\mathscr{A} - 6}{\mathscr{A} + 12}$$

Bjorken flow
in MIS theory



There is a unique solution regular at the small w, which acts as a universal attractor for generic solutions.

Bjorken flow in MIS theory

asymptotics

The late-time asymptotic behaviour of solutions of

$$C_\tau \left(1 + \frac{\mathcal{A}}{12}\right) \mathcal{A}' + \frac{C_\tau}{3w}\mathcal{A}^2 = \frac{3}{2}\left(\frac{8C_\eta}{w} - \mathcal{A}\right)$$

is universal up to exponential corrections:

Calculable constants

$$\mathcal{A} = \underbrace{\frac{8C_\eta}{w}}_{\text{Navier–Stokes}} + \underbrace{\frac{16C_\eta C_\tau}{3w^2}}_{\text{2nd order}} + \ldots = \underbrace{\sum_{n>0} \frac{a_n^{(0)}}{w^n}}_{\text{gradient expansion}} + \underbrace{\left(\sigma\, w^{\frac{C_\eta}{C_\tau}} e^{-\frac{3}{2C_\tau}w}\right) \sum_{n\geq 0} \frac{a_n^{(1)}}{w^n}}_{\text{transseries sectors}} + \ldots$$

The transseries describes the dissipation of information contained in the initial state, carried by the trans series parameter σ.

The damping rate is set by the nonhydrodynamic mode frequency.

Bjorken flow in MIS
asymptotics

Asymptotic behaviour of solutions of

$$C_\tau\left(1+\frac{\mathscr{A}}{12}\right)\mathscr{A}'+\frac{C_\tau}{3w}\mathscr{A}^2=\frac{3}{2}\left(\frac{8C_\eta}{w}-\mathscr{A}\right)$$

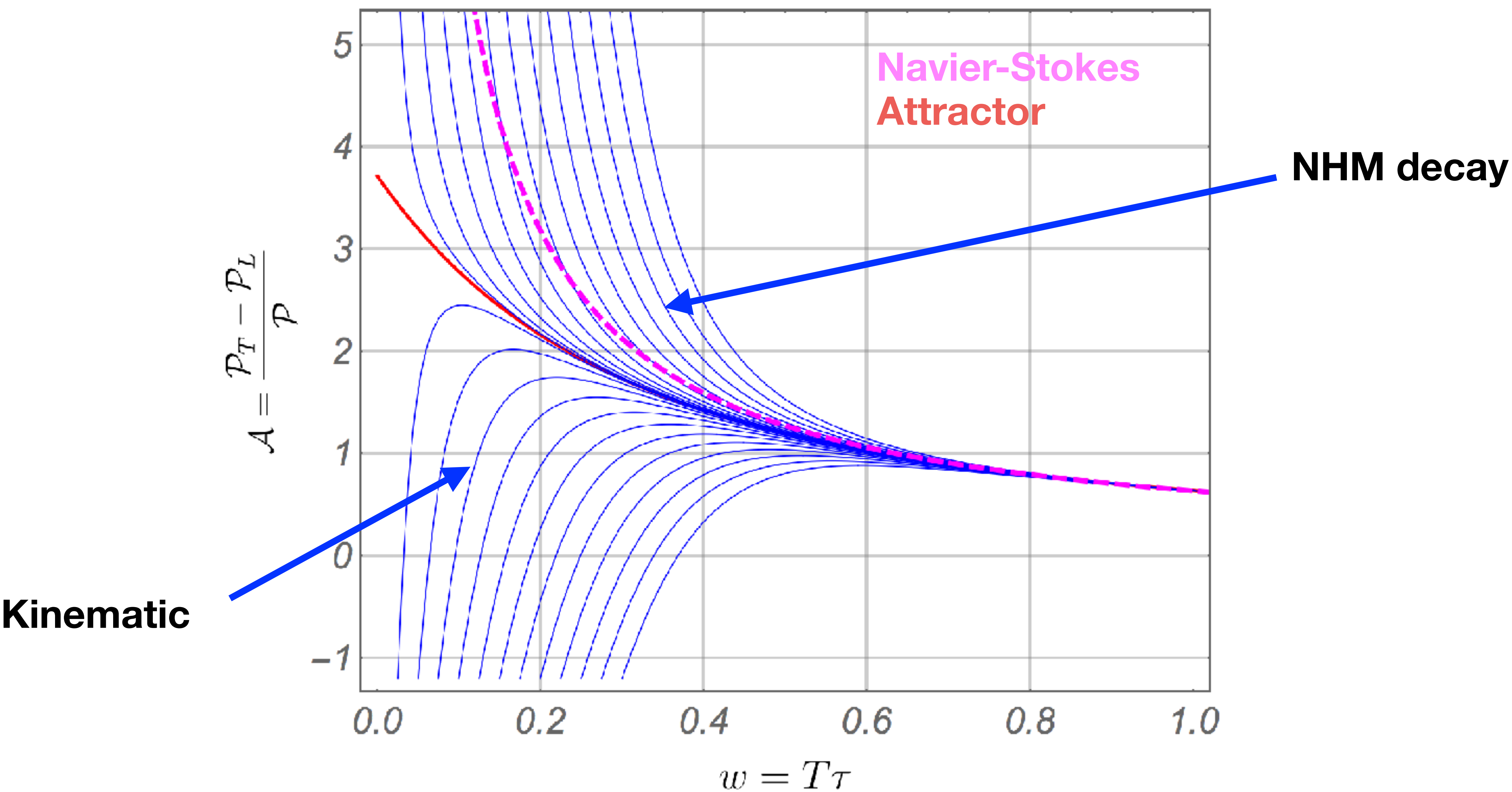
at early times:

$$\mathscr{A}=\pm 6\sqrt{\frac{C_\eta}{C_{\tau\Pi}}}+O(w) \quad \textbf{or} \quad \mathscr{A}\sim 1/w^4$$

The upper sign give the attractive" solution.

- The attractor depends on the transport coefficients.
- The collapse toward the attractor at early times is independent of transport coefficients: this suggests it is a kinematic effect

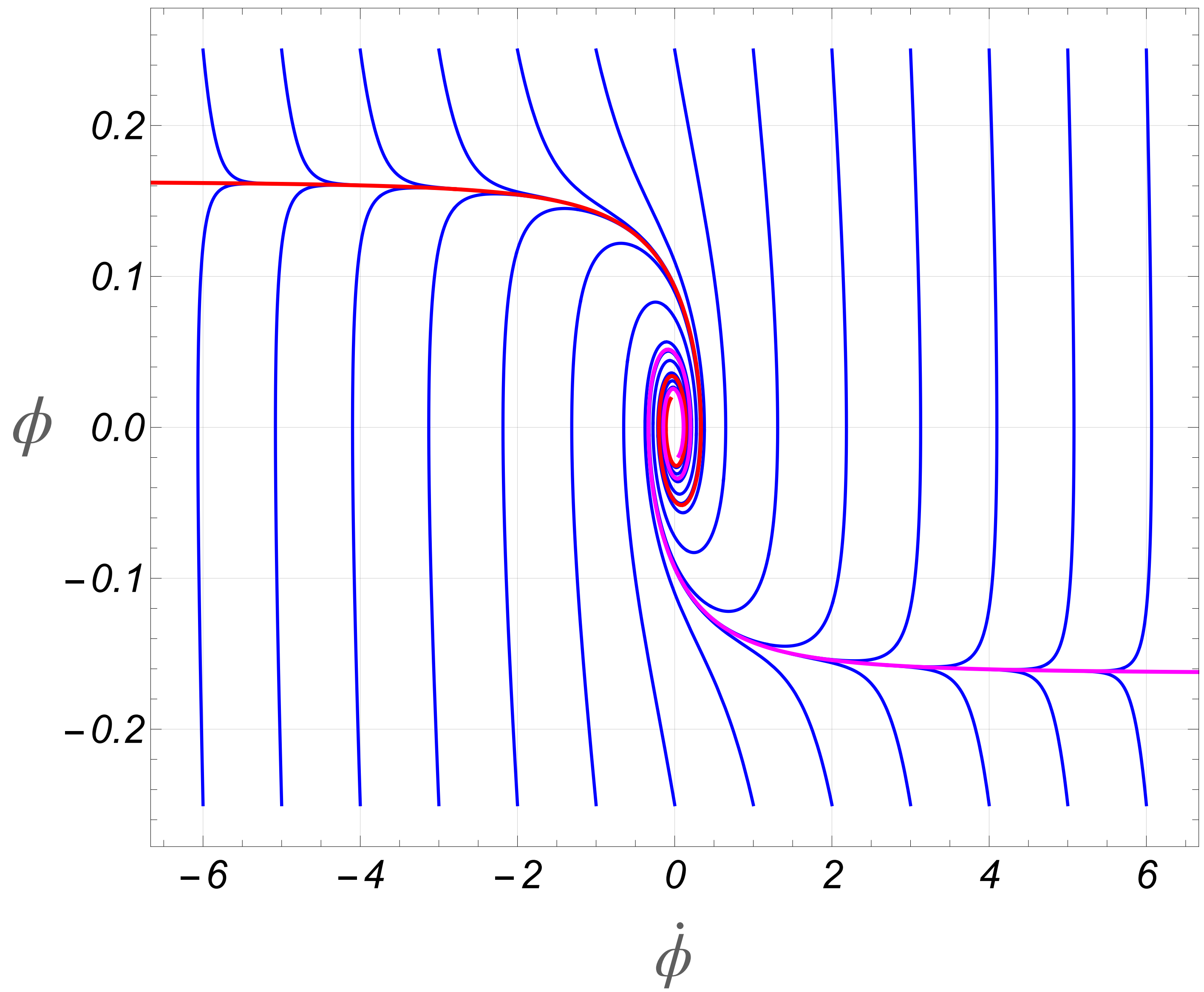
Bjorken flow
in MIS theory



$$\mathscr{E}(\tau) \sim \frac{\mu^4}{(\mu\tau)^\beta} \iff \mathscr{A}(w) \sim 6(1 - \frac{3}{4}\beta)$$

At early times

An example of attractor behaviour
in inflationary cosmology



A wide class of initial conditions quickly end up on the inflationary attractor, where the potential gradient competes with the Hubble expansion of the gravitational background.

See e.g. Mukhanov's "Physical Cosmology", Cambridge University Press 2005

Toward more complex models
MIS theory in a general frame

If we relax the Landau frame condition

$$\Pi^{\mu\nu} = \pi^{\mu\nu} + \frac{1}{3}\mathscr{B}\left(\eta^{\mu\nu} + 4u^{\mu}u^{\nu}\right) + \mathscr{Q}^{\mu}u^{\nu} + \mathscr{Q}^{\nu}u^{\mu}$$

we need to prescribe equations for all the dissipative fields.

One option is NS theory in a general frame – BDNK.

Another option is MIS theory in a general frame, where we posit relaxation equations for all the dissipative fields:

$$\mathscr{B} = -\tau_B \mathscr{D}\mathscr{B} - \tau_\varphi \mathscr{D}\mathscr{E}$$

$$\mathscr{Q}^{\mu} = -\tau_Q \mathscr{D}\mathscr{Q}^{\mu} - \tau_\psi \Delta^{\mu\lambda}\mathscr{D}_\lambda \mathscr{E}$$

$$\pi^{\mu\nu} = -\tau_\pi \mathscr{D}\pi^{\mu\nu} - 2\eta\sigma^{\mu\nu}$$

Having more degrees of freedom (initial conditions) allows encoding more information about the initial state.

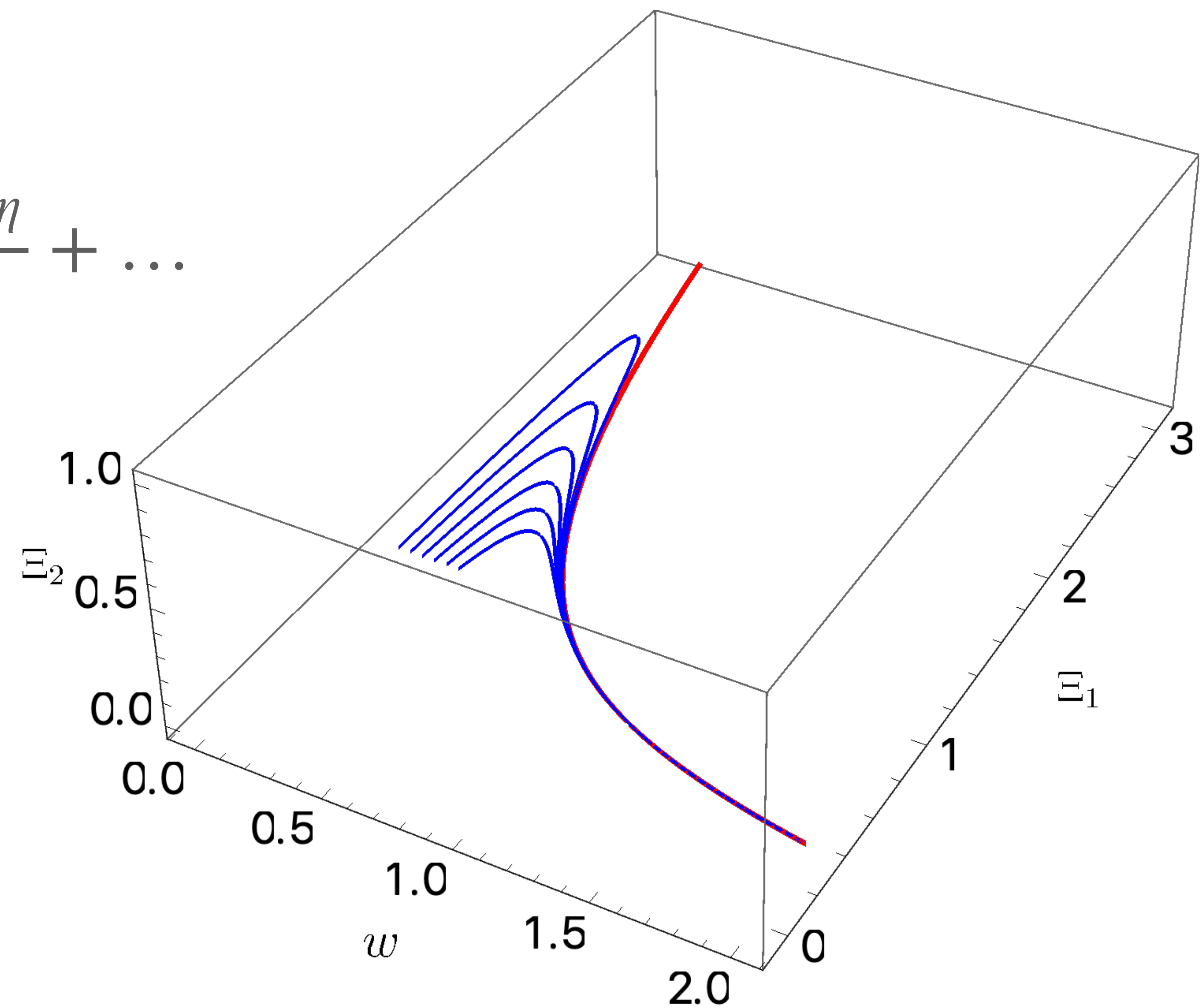
MIS theory in a general frame
the hydrodynamic attractor

Here the phase space has an extra dimension compared to MIS.

Dimensionless variables:

$$\Xi_1 \equiv 6\left(1 + \frac{3}{4}\tau\partial_\tau \ln \mathscr{E}\right) \sim \frac{8C_\eta}{w} + \dots$$

$$\Xi_2 \equiv \frac{\mathscr{B}}{\mathscr{E}} \sim -\frac{63C_\eta C_\varphi}{27w^2} + \dots$$



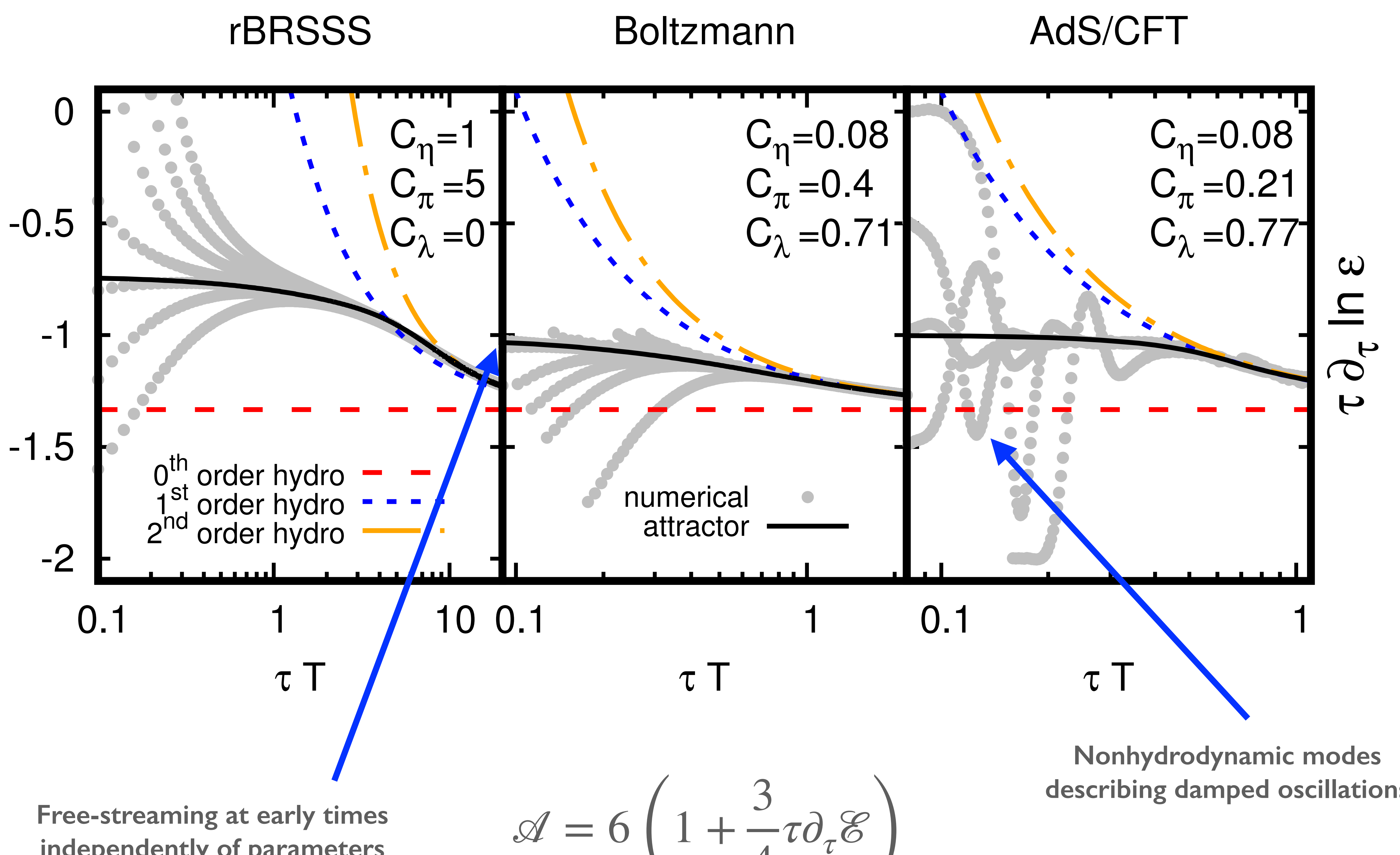
Pressure anisotropy

$$\mathscr{A} \sim \frac{8C_\eta}{w} - \frac{16C_\eta C_\pi}{3w^2} + \dots$$

Dependence on C_φ, C_B enters at higher orders.

Hydrodynamic attractors

as a generic phenomenon in Bjorken flow



$$\mathscr{A} = 6\left(1 + \frac{3}{4}\tau\partial_\tau \mathscr{E}\right)$$

Romatchke 1704.08699

Equilibration in N=4 Super Yang-Mills theory

using holography

The AdS/CFT correspondence identifies thermal states of strongly coupled N=4 SYM in 4-d Minkowski space with black objects in 5d asymptotically-AdS gravity solutions.

Black hole quasinormal modes (damped oscillations of horizon) map to linear perturbations of thermal equilibrium.

There is an infinite number of nonhydrodynamic QNM, whose frequencies come in complex-conjugate pairs.

Numerical studies of boost-invariant dynamical black hole spacetimes provide a picture of equilibration matching low orders of the gradient expansion at anisotropy of order one.

Romatschke's attractor can be partly reproduced by Borel summation of the gradient expansion.

Modeling equilibration in SYM

with HJSW hydrodynamics

Replace the MIS relaxation equation by

$$\left(\left(\frac{1}{T}\mathscr{D} \right)^2 + 2\Omega_I \frac{1}{T}\mathscr{D} + |\Omega|^2 \right) \Pi^{\mu\nu} = -\eta |\Omega|^2 \sigma^{\mu\nu} - c_\sigma \frac{1}{T}\mathscr{D}\left(\eta\sigma^{\mu\nu}\right) + \dots$$

where $\Omega \equiv \Omega_R + i\Omega_I$.

Linearised perturbations of equilibrium reveal a pair of nonhydrodynamic modes with conjugate frequencies

$$\left. \frac{1}{T}\omega \right|_{k=0} = \pm\,\Omega_R - i\,\Omega_I$$

Time evolution is stable and causal at the linearised level for suitably chosen parameters, although the SYM values for the QNM frquencies lie outside this region.

Modeling equilibration in SYM

with Bjorken flow

For Bjorken flow

$$\alpha_1 \mathscr{A}'' + \alpha_2 \mathscr{A}'^2 + \alpha_3 \mathscr{A}' + 12\,\mathscr{A}^3 + \alpha_4\,\mathscr{A}^2 + \alpha_5\,\mathscr{A} + \alpha_6 = 0$$

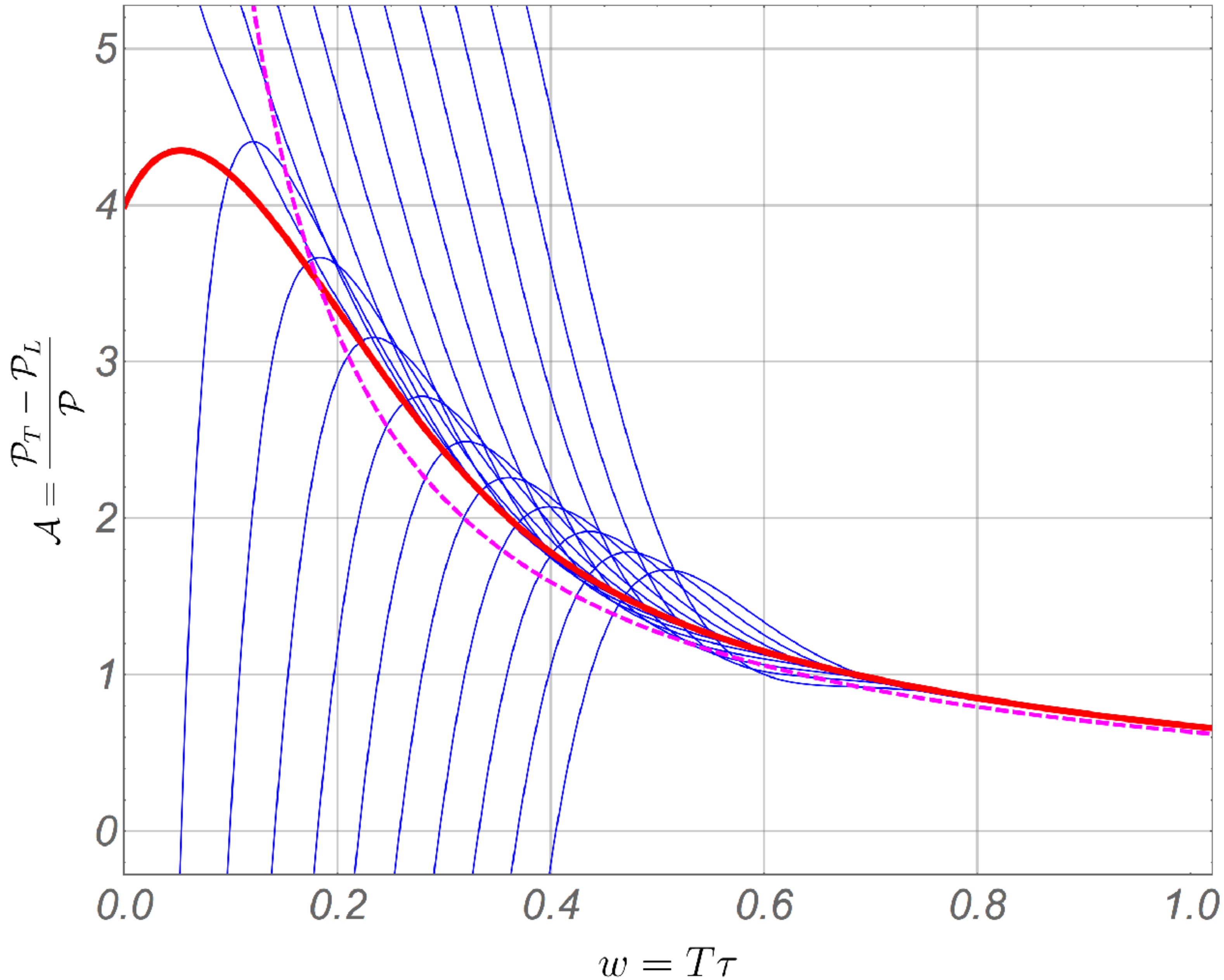
where

$$\begin{aligned}
\alpha_1 &= w^2\,(\mathscr{A} + 12)^2, \\
\alpha_2 &= w^2\,(\mathscr{A} + 12), \\
\alpha_3 &= 12\,w\,(\mathscr{A} + 12)\,(\mathscr{A} + 3\,w\,\Omega_I), \\
\alpha_4 &= 48\,(3\,w\,\Omega_I - 1), \\
\alpha_5 &= 108\,\left(-4\,C_\eta\,C_\sigma + 3\,w^2\,\Omega^2\right), \\
\alpha_6 &= -\,864\,C_\eta\,\left(-2\,C_\sigma + 3\,w\,\Omega^2\right)
\end{aligned}$$

The phase space for Bjorken flow is 3-dimensional here.

Modeling equilibration in SYM

the attractor of HJSW hydrodynamics



At late times

$$\mathscr{A}(w) \sim 8\, C_\eta \frac{1}{w} + \frac{16\, C_\eta (-C_\sigma + 2\,\Omega_I)}{3\,|\Omega|^2} \frac{1}{w^2} + \ldots$$

Divergence of the gradient expansion
and universality near equilibrium

Consider the expectation value of the energy momentum tensor in some microscopic theory. Close to equilibrium we will find

$$\langle \hat{T}^{\mu\nu} \rangle = \mathcal{E} u^\mu u^\nu + \mathcal{P}(\mathcal{E})(g^{\mu\nu} + u^\mu u^\nu) + \eta \sigma^{\mu\nu} + \ldots$$

What do we expect?

- Option 1: for gradients of fixed magnitude, adding more terms will give an increasingly more accurate answer

- Option 2: for a fixed number of terms, the answer will become more accurate as the magnitude of the gradients diminishes

The second possibility means that the series is asymptotic but not necessarily convergent.

At late times: universal asymptotic behaviour across many theories.

Divergence of the gradient expansion

and universality near equilibrium

Examples where many coefficients are known

- Hydrodynamic models
- N=4 SYM via AdS/CFT
- Kinetic Theory (RTA)

The divergence of the gradient expansion:

- expresses the fact that subdominant contributions had been dropped
- explains why hydrodynamics works so well: “divergent series converge faster than convergent series” (G. Carrier)
- is connected with nonhydrodynamic modes of the theory so it should be seen as generic.

Attractors by Borel summation

The gradient expansion can be calculated in many models:

$$\mathscr{A}(w) = \frac{8C_\eta}{w} + \ldots = \sum_{n>0} \frac{a_n^{(0)}}{w^n}$$

Often hundreds of terms are available.

These asymptotic series are almost always divergent, but can be resumed using the Borel summation formula originating from

$$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} c_n \underbrace{\left(\frac{1}{n!} \int_0^\infty t^n e^{-t} dt \right)}_{1} = \int_0^\infty \underbrace{\left(\sum_{n=0}^{\infty} \frac{c_n}{n!} \right)}_{\text{Borel transform}} t^n e^{-t} dt$$

This provides a practical avenue for estimating the attractor for a significant range of w.

Attractors by Borel summation

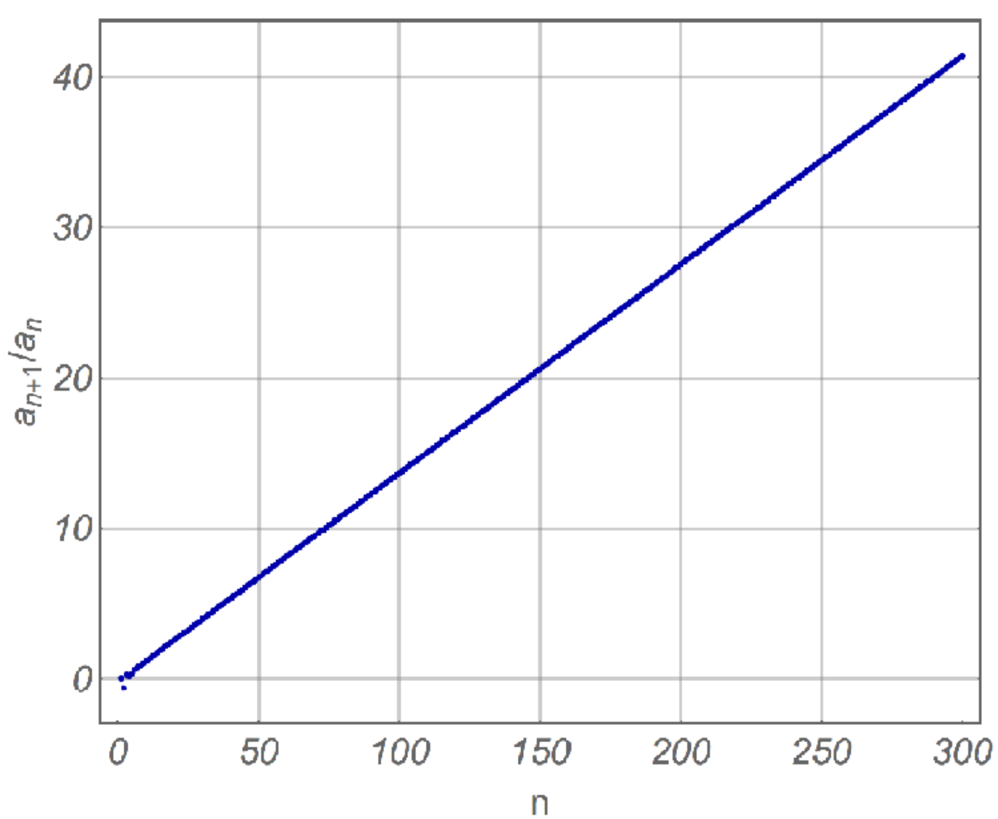
for Bjorken flow in MIS

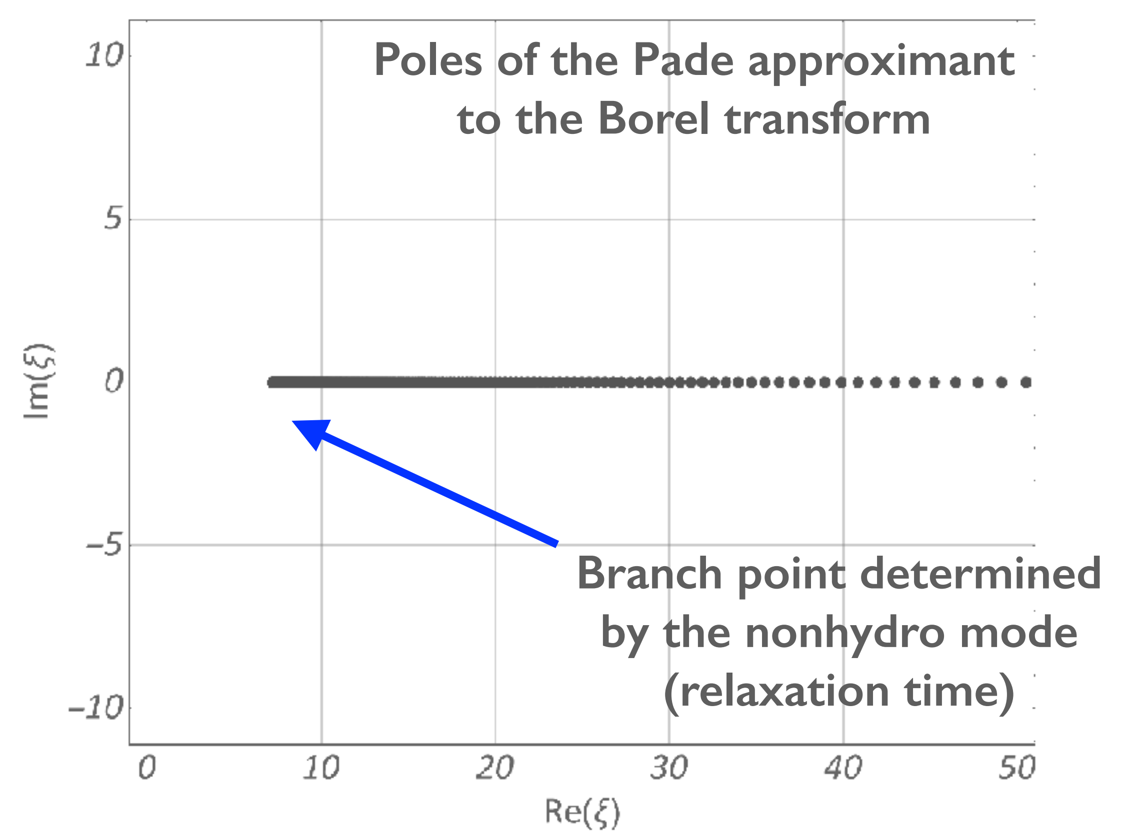
$$\mathscr{A} = \underbrace{\frac{8C_\eta}{w}}_{\text{Navier–Stokes}} + \underbrace{\frac{16C_\eta C_\tau}{3w^2}}_{\text{2nd order}} + \ldots = \underbrace{\sum_{n>0} \frac{a_n^{(0)}}{w^n}}_{\text{gradient expansion}} + \underbrace{\left(\sigma\, w^{\frac{C_\eta}{C_\tau}} e^{-\frac{3}{2C_\tau}w}\right) \sum_{n\geq 0} \frac{a_n^{(1)}}{w^n}}_{\text{transseries sectors}} + \ldots$$

The Borel transform

$$\mathscr{B}[\sum_{n\geq 0} \frac{a_{n+1}^{(0)}}{w^n}](\xi) = \sum_{n\geq 0} \frac{a_{n+1}^{(0)}}{\Gamma[n+1]} \xi^n$$

The series are truncated and Pade approximates are used to analytically continue the Borel transform





Attractors by Borel summation

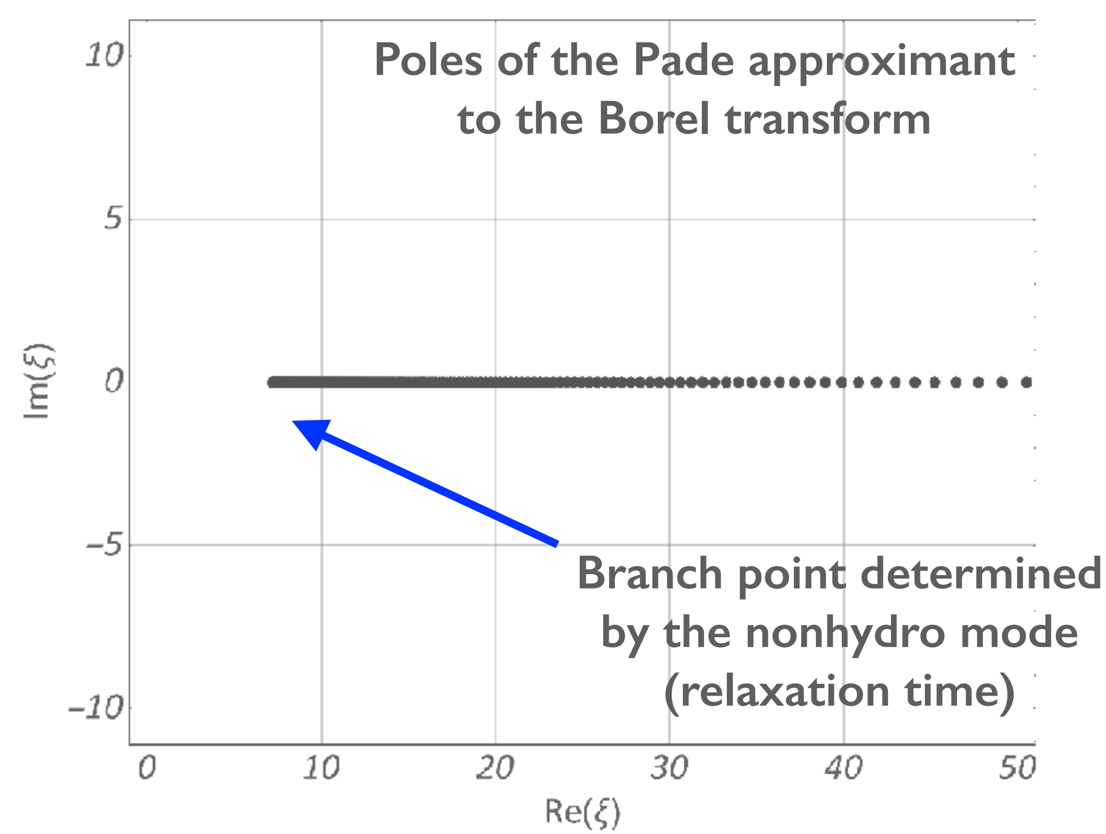
for Bjorken flow in MIS

$$\mathscr{A} = \underbrace{\frac{8C_\eta}{w}}_{\text{Navier–Stokes}} + \underbrace{\frac{16C_\eta C_\tau}{3w^2}}_{\text{2nd order}} + \ldots = \underbrace{\sum_{n>0}\frac{a_n^{(0)}}{w^n}}_{\text{gradient expansion}} + \underbrace{\left(\sigma\, w^{\frac{C_\eta}{C_\tau}}\, e^{-\frac{3}{2C_\tau}w}\right)\sum_{n\geq 0}\frac{a_n^{(1)}}{w^n}}_{\text{transseries sectors}} + \ldots$$

$$\mathscr{B}\Big[\sum_{n\geq 0}\frac{a_{n+1}^{(0)}}{w^n}\Big](\xi) = \sum_{n\geq 0}\frac{a_{n+1}^{(0)}}{\Gamma[n+1]}\xi^n$$

Pade approximant

$$\mathcal{S}\Big[\sum_{n\geq 0}\frac{a_{n+1}^{(0)}}{w^n}\Big] = \int_C d\xi\, e^{-w\xi}\, \tilde{\mathscr{B}}\Big[\sum_{n\geq 0}\frac{a_{n+1}^{(0)}}{w^n}\Big](\xi)$$



The Borel sum reproduces the attractor (approximately!) down to moderate values of w

Attractors by Borel summation

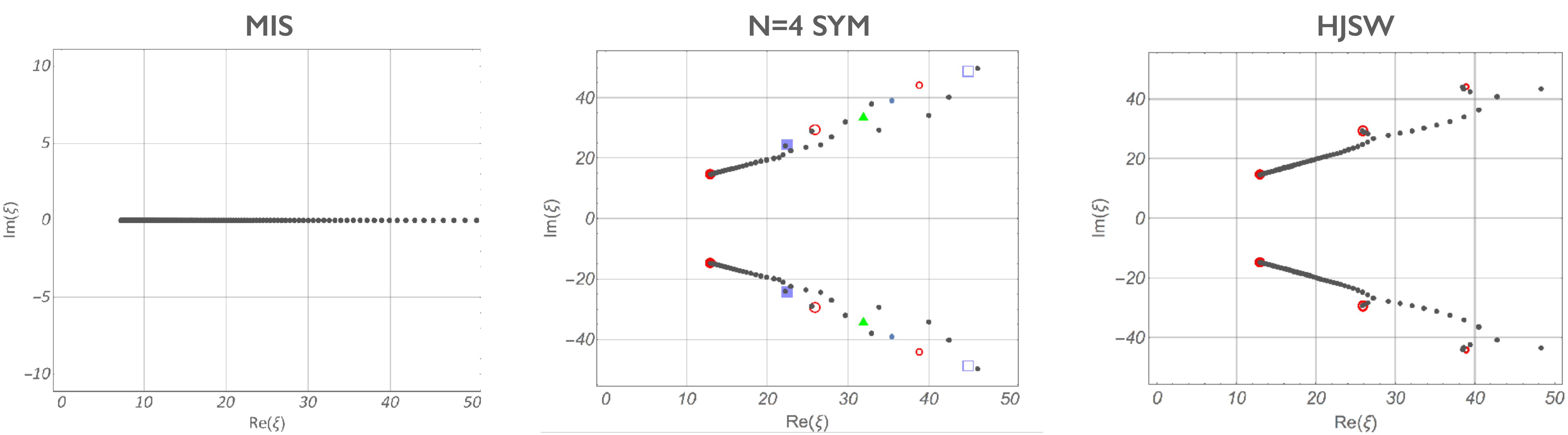
for Bjorken flow in

When the expansion coefficients are available we can use

$$\mathscr{B}[\sum_{n\geq 0} \frac{a_{n+1}^{(0)}}{w^n}](\xi) = \sum_{n\geq 0} \frac{a_{n+1}^{(0)}}{\Gamma[n+1]}\xi^n$$

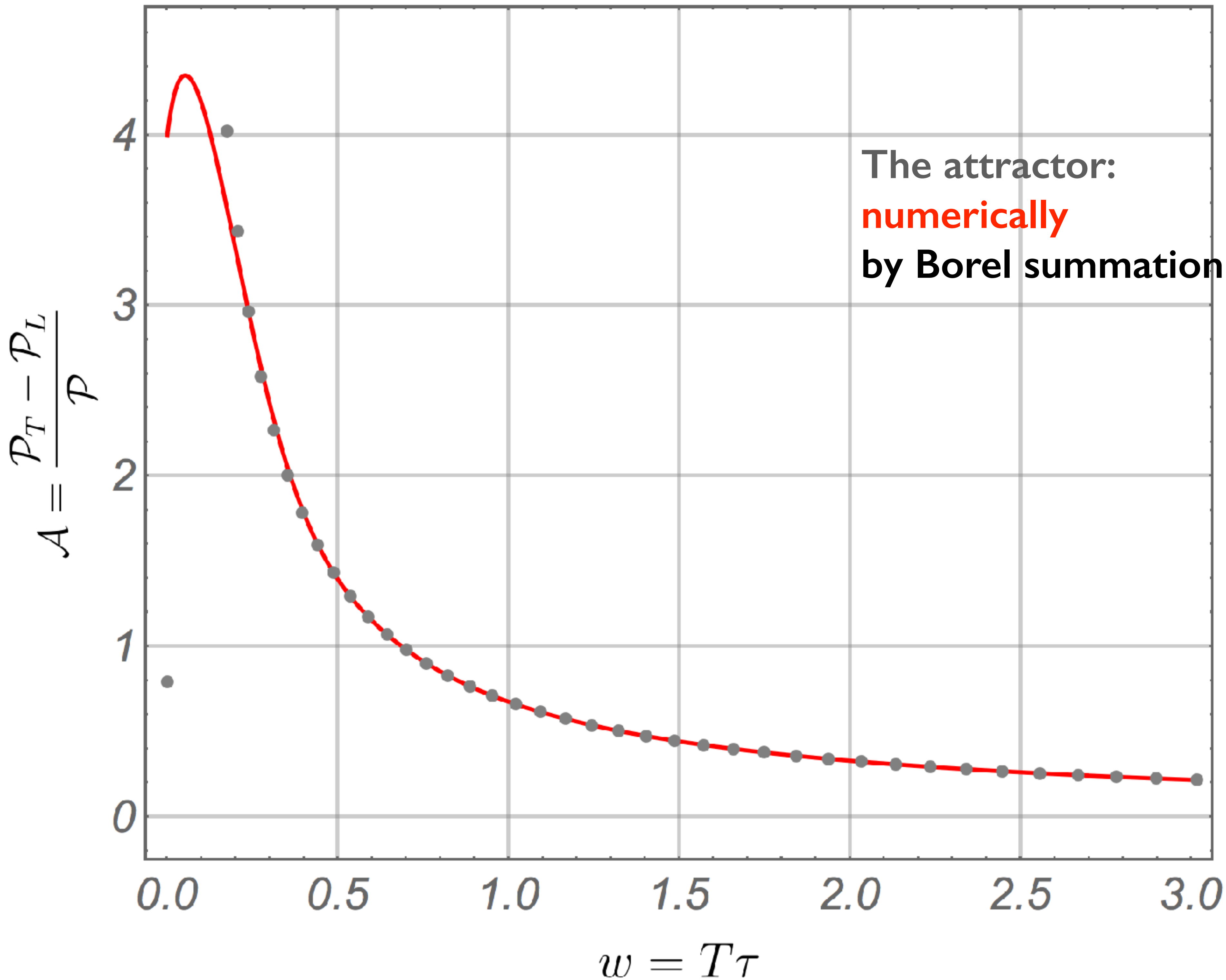
$$\mathcal{S}[\sum_{n\geq 0} \frac{a_{n+1}^{(0)}}{w^n}] = \int_C d\xi\, e^{-w\xi}\, \tilde{\mathscr{B}}[\sum_{n\geq 0} \frac{a_{n+1}^{(0)}}{w^n}](\xi)$$

In the case of SYM/HJSW there is no complex ambiguity



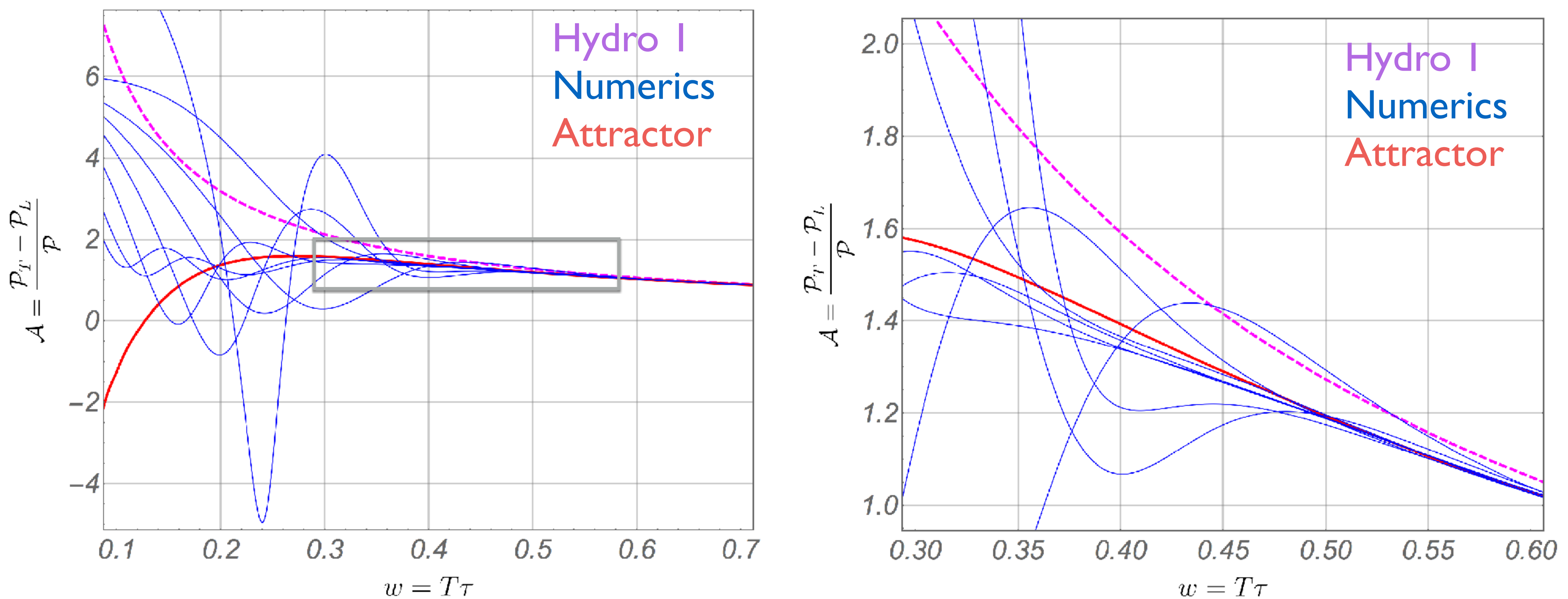
Attractors by Borel summation

of the gradient expansion in HJSW



Attractors by Borel summation

of the gradient expansion in N=4 SYM



- Resummation matches Romatchke's attractor down to $w \approx 0.45$
- Initial states set by a functions of the radial AdS coordinate
- Initially $\mathscr{A} = 6$ for all solutions in these plots

The Phase Space Approach
in generic variables

Until now we relied on the special parametrisation of phase space in terms of $(w, \mathscr{A})$. This is hard to generalise, so we will now try to identify attractors using generic variables.

The basic idea can be illustrated using Bjorken flow. In any theory the late time behaviour is

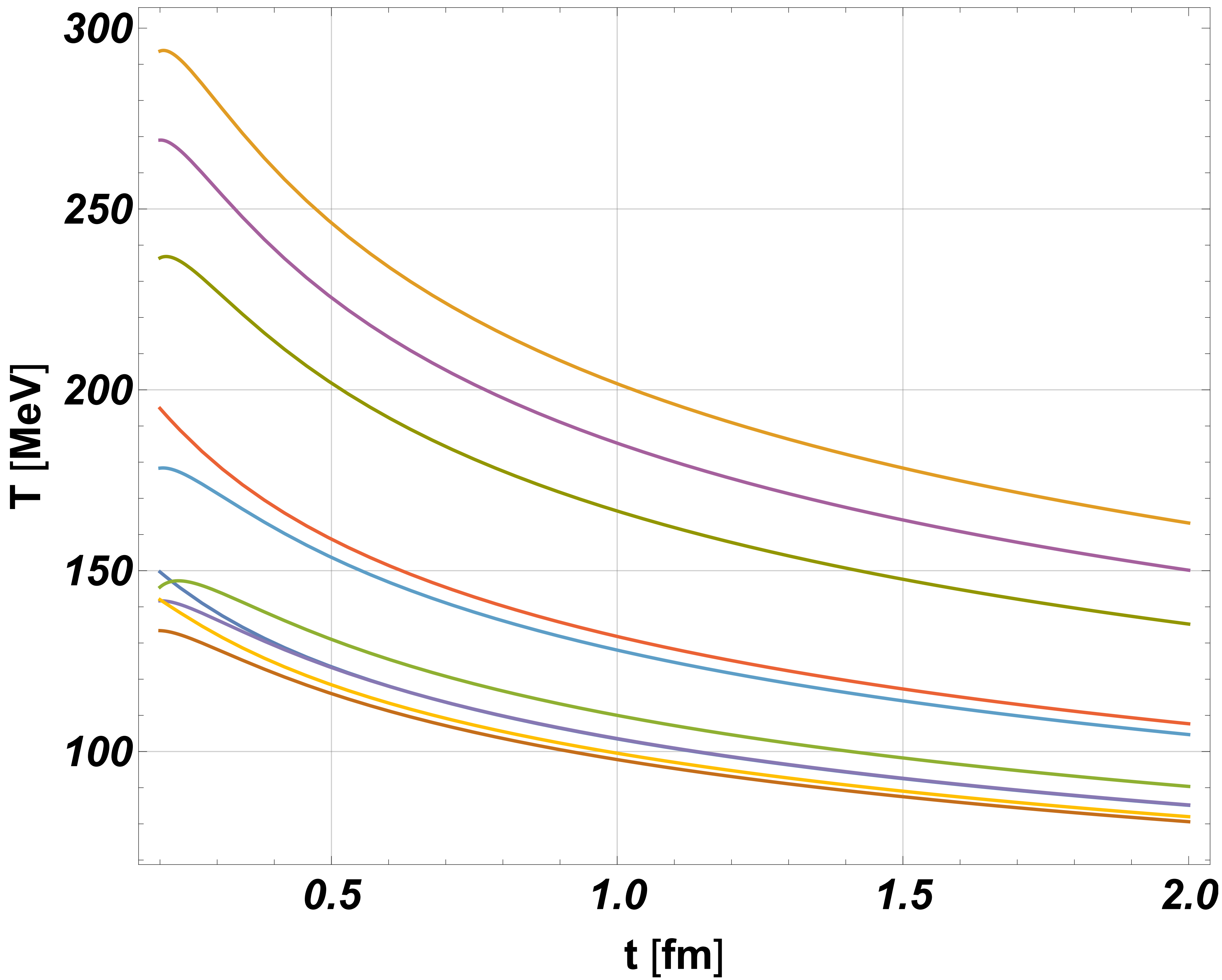
$$T(\tau) \sim \frac{\Lambda}{(\Lambda\tau)^{1/3}}$$

where the scale Λ is the only remnant of the initial data which survives until late times. A region on some initial time slice of phase space is reduced to a one dimensional locus at late times.

This identifies the hydrodynamic attractor with dimensionality reduction of phase space regions.

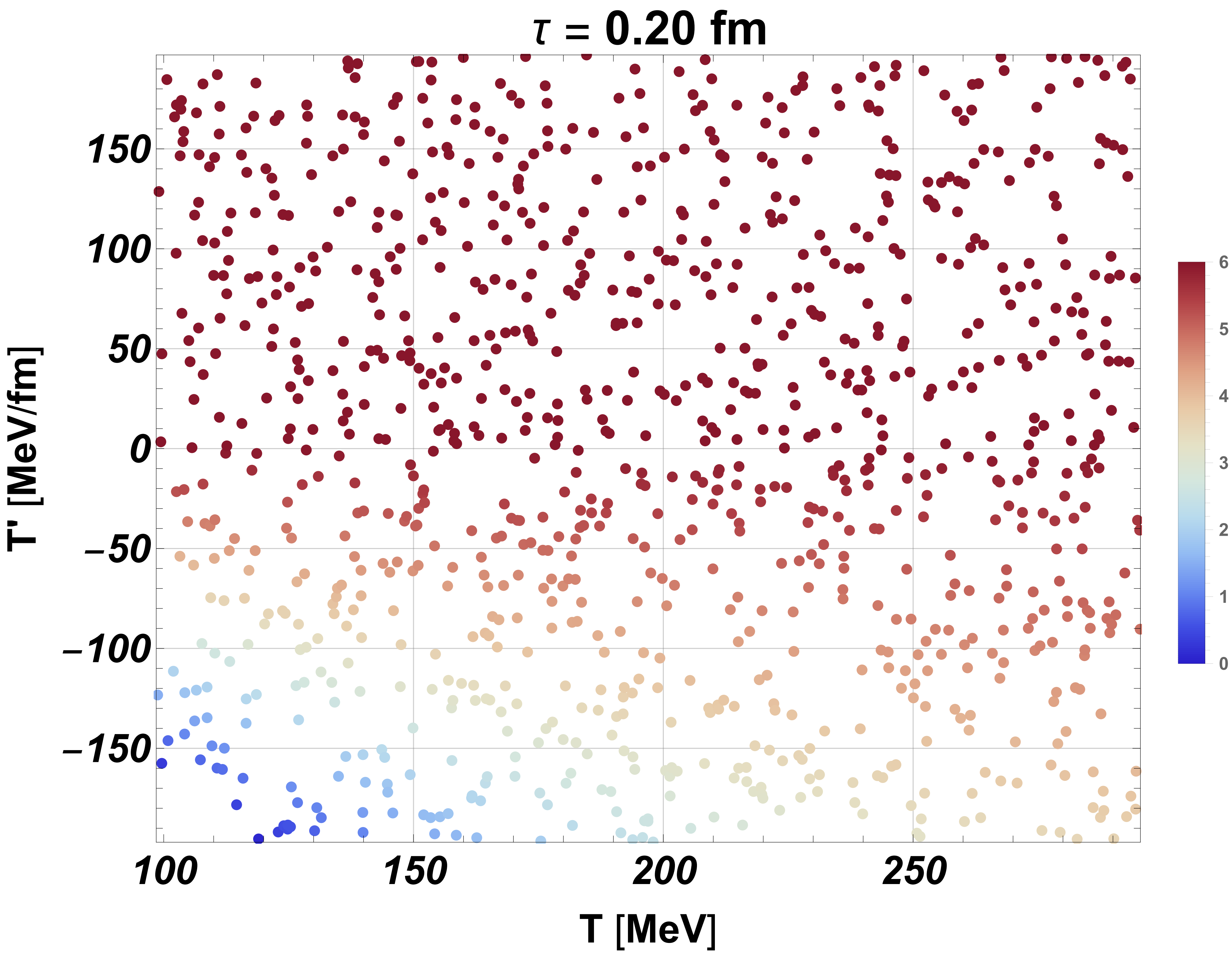
The Phase Space Approach

Can you see the MIS attractor in this picture?



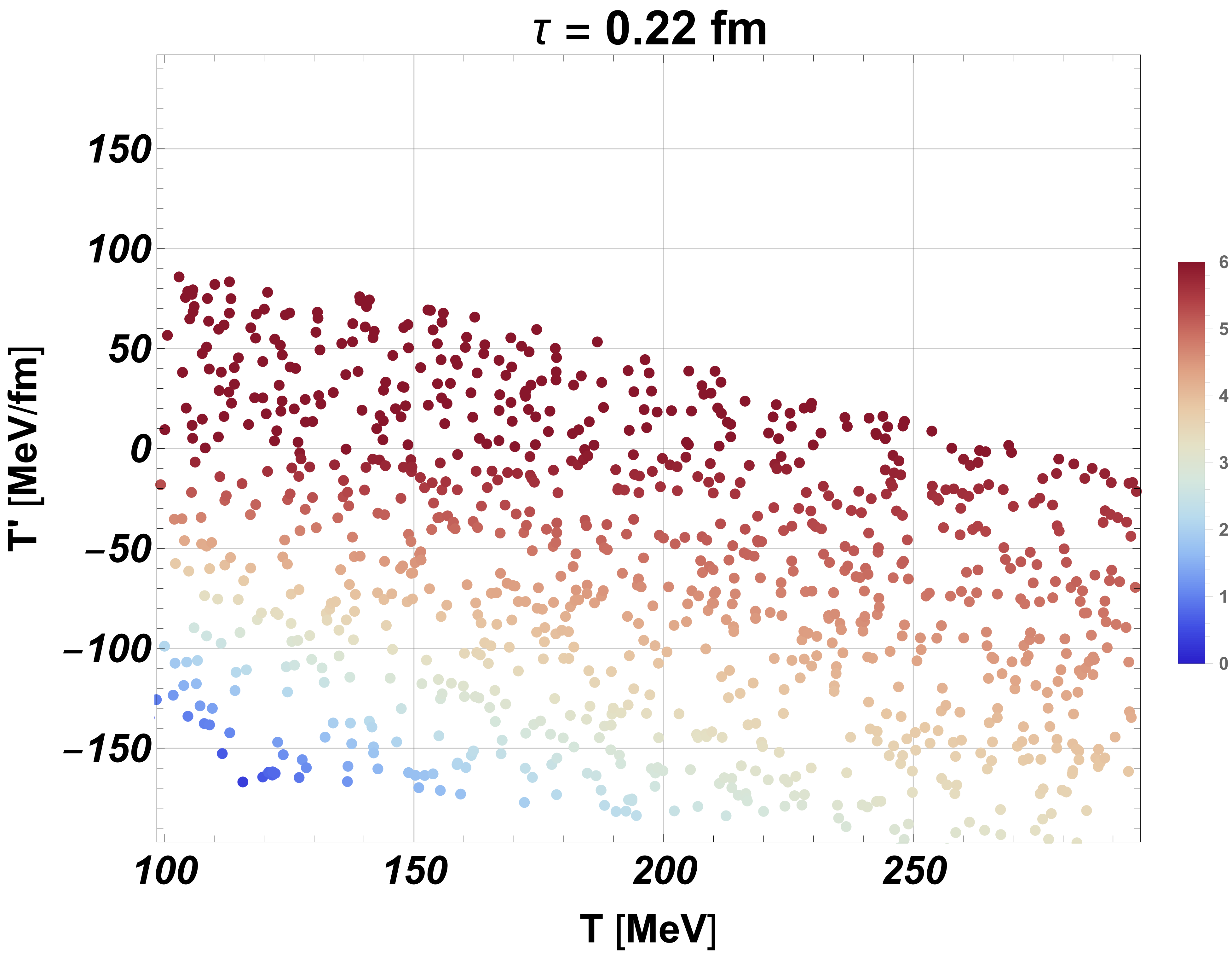
The hydrodynamic attractor in MIS

A view from phase space



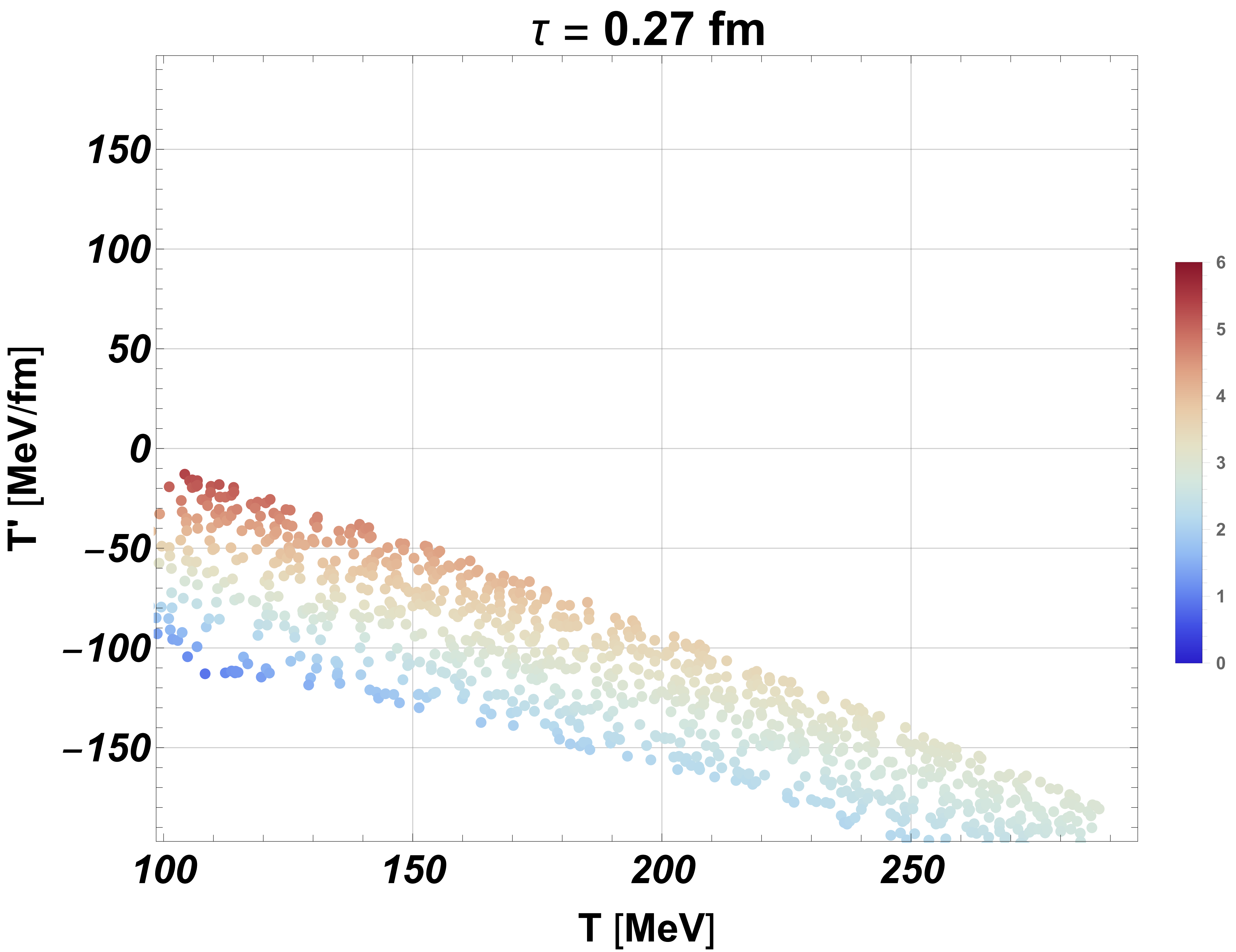
The hydrodynamic attractor in MIS

A view from phase space



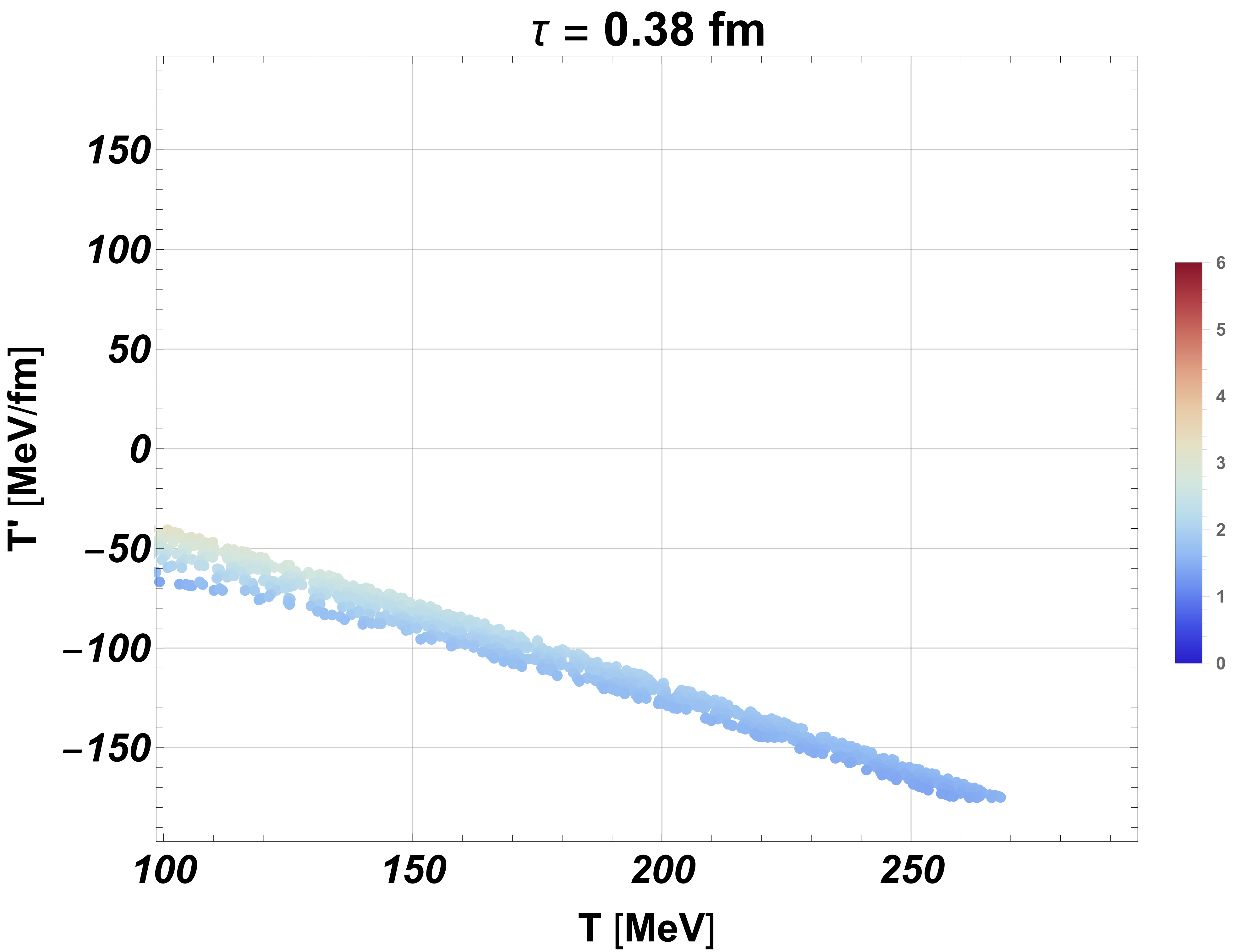
The hydrodynamic attractor in MIS

A view from phase space



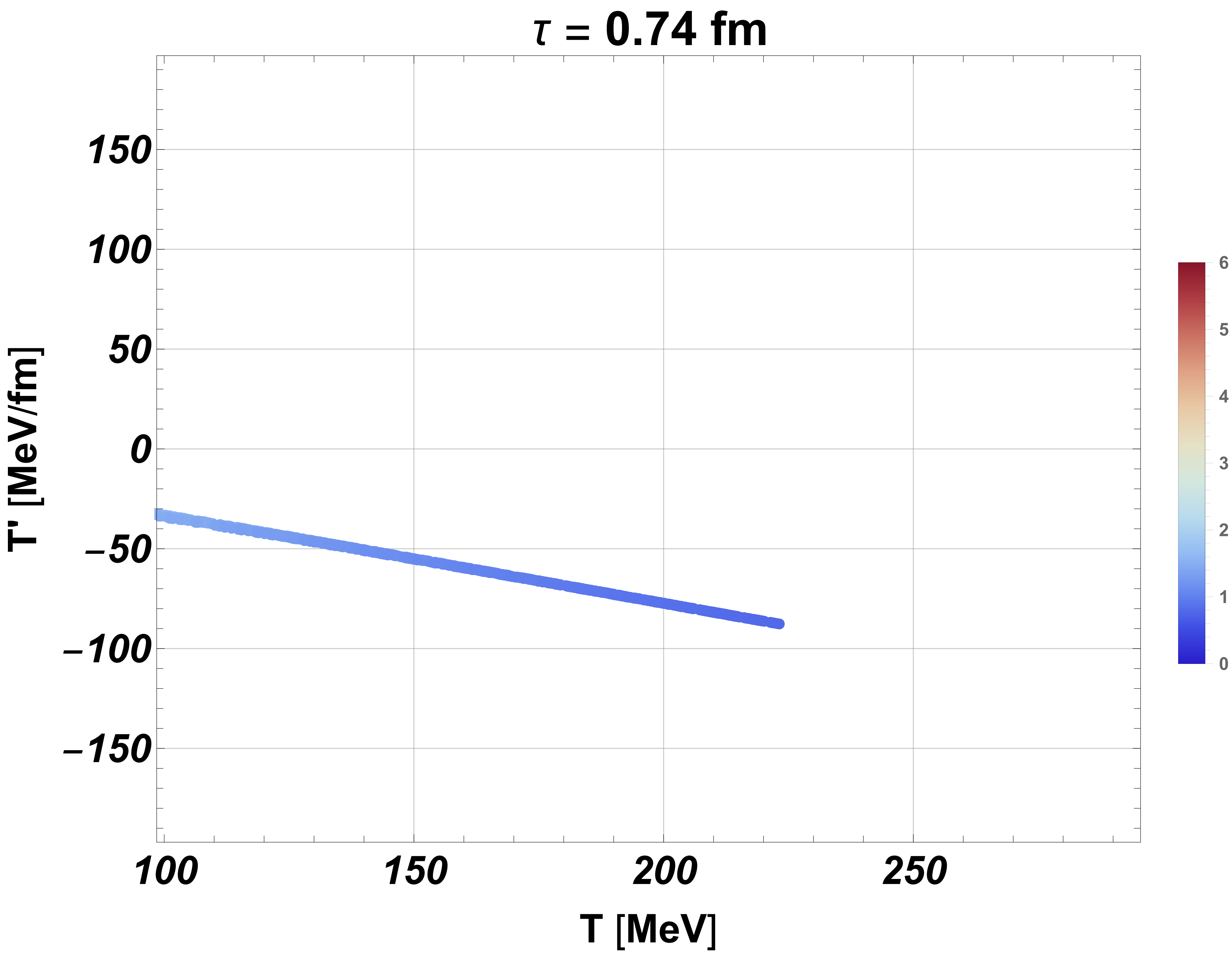
The hydrodynamic attractor in MIS

A view from phase space



The hydrodynamic attractor in MIS

A view from phase space



The Phase Space Approach

A set of solutions spanning a D-dimensional region on the initial time slice ends up in a region of lower dimensionality $d<D$ on a subsequent time slice.

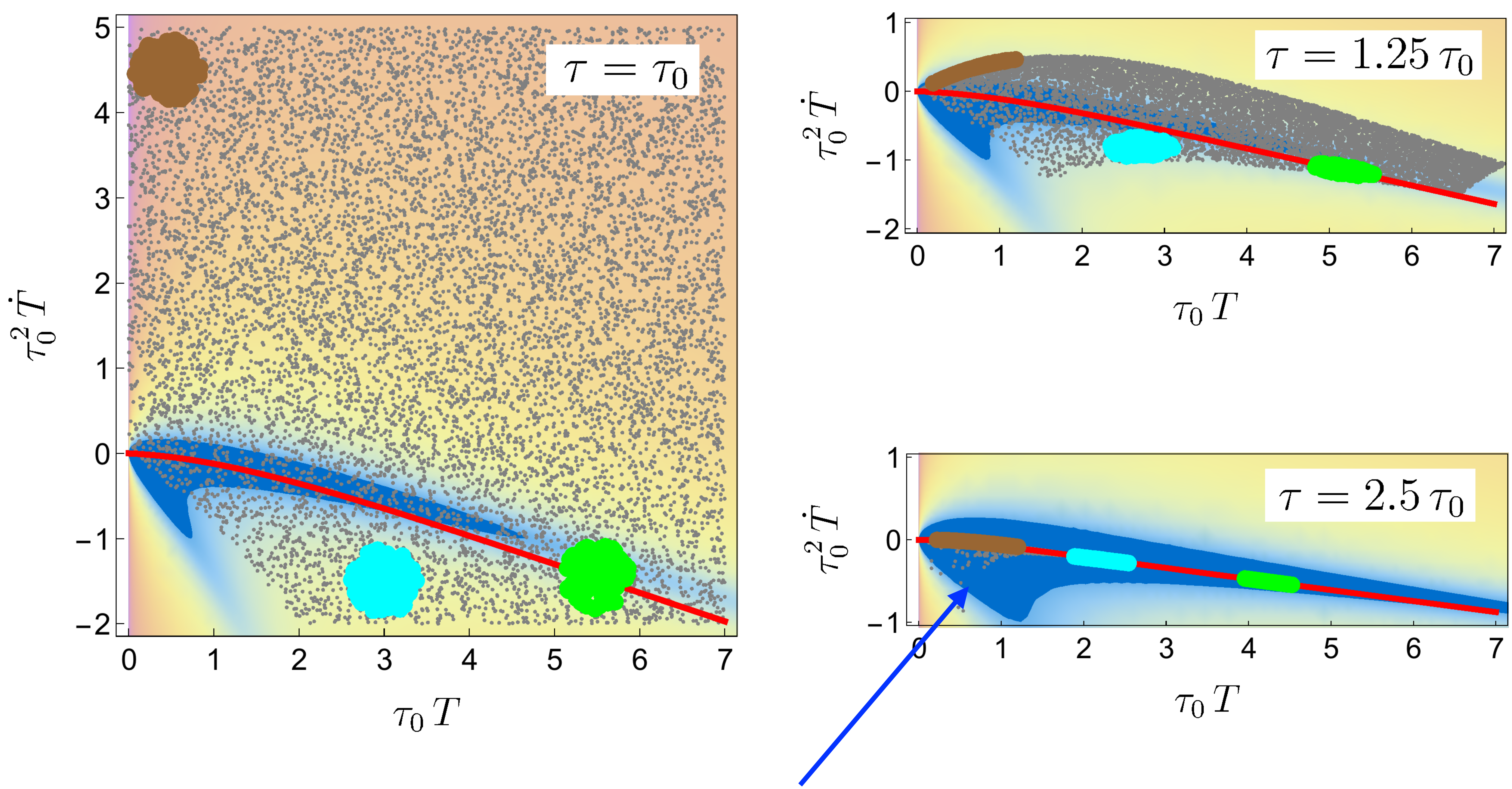
The attractor phenomenon is identified with this reduction of dimensionality of sets of solutions on slices of phase space.

- No special variables are necessary
- No limitations such as Bjorken symmetry or conformal symmetry
- Natural application area for techniques of data science/ML

Only some pilot studies so far.

The Phase Space Approach

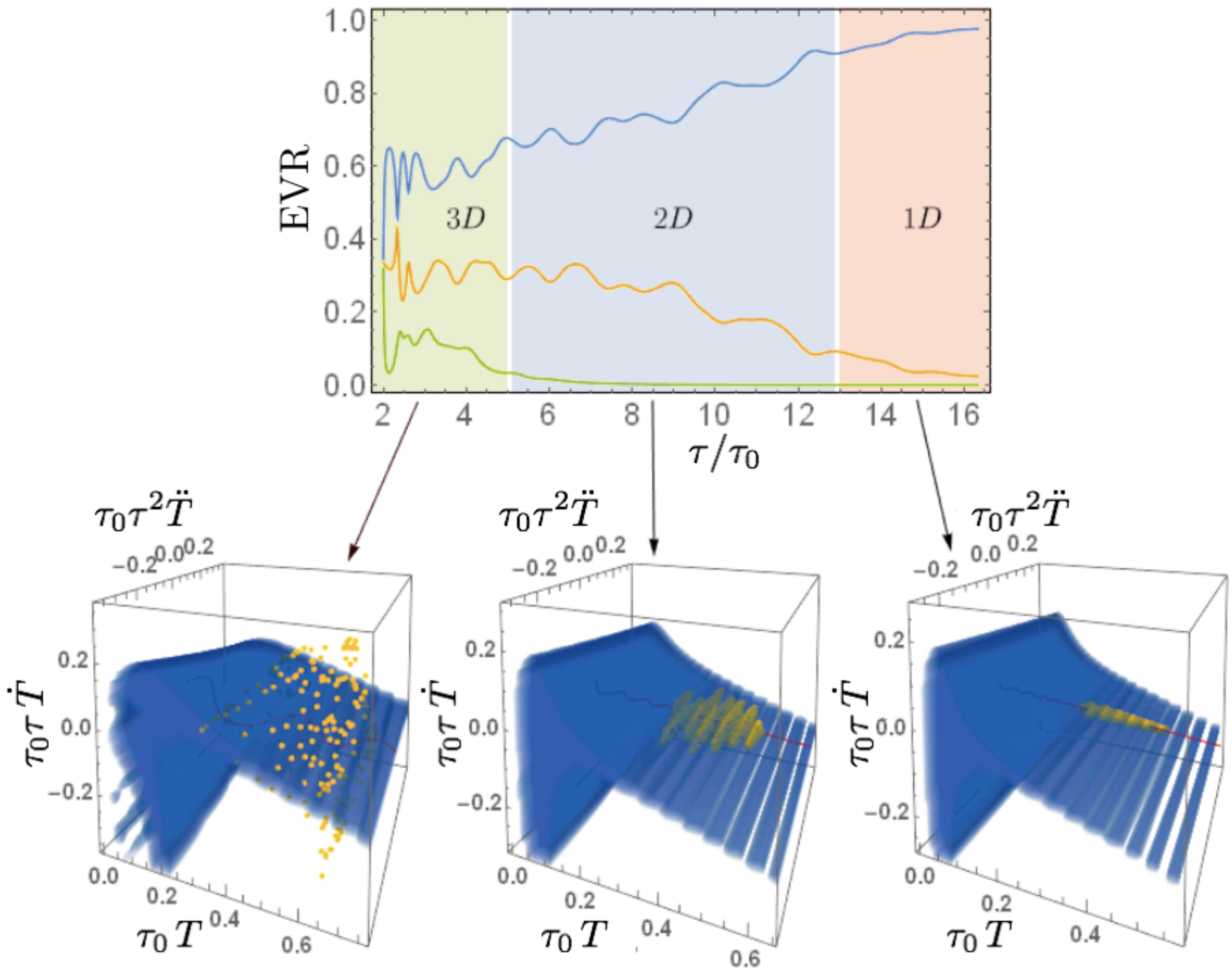
Bjorken flow in MIS theory



"Slow region"

The Phase Space Approach

Bjorken flow in HJSW



Summary/Outlook

- In toy models, the effectiveness of fluid dynamics at early times could be explained using the concept of hydrodynamic attractors.
- The origin of attractor behaviour at early times has been linked to the longitudinal expansion and as such may be of kinematic nature.
- This may also occur in QCD given the same kinematics, so hydrodynamic models should try to mimic the QCD attractor.
- Recent numerical results suggest that early-time hydrodynamic attractors survive the inclusion of transverse dynamics as well as the breaking of conformal symmetry.

Kurkela et al. 1907.08101; Ambrus et al. 2102.11785, 2109.03290.
Chattopadhyay et al. 2107.05500; Jaiswal et al. 2107.10248; Kamata et al. 2206.00653.