Fluid Dynamics and Kinetic theory

1. Basic elements of fluid dynamics

- Basic elements: energy and momentum density and fluxes

$\longleftrightarrow$ Energy-momentum tensor $T^{\mu\nu}$:

T^{00} energy density
T^{0i} $i=(x,y,z)$ momentum density
T^{i0} energy flux in i-direction (or flux density)
T^{ij} j-momentum flux in i-direction

- Energy conservation: $\partial_t T^{00} = -\partial_x T^{x0} - \partial_y T^{y0} - \partial_z T^{z0}$

infinitesimal cube: $\longrightarrow$ 

$$\underbrace{\partial_t \int T^{00}\, dV}_{\text{change of } E \text{ within the cube}} = - \underbrace{\oint T^{i0}\, dA_i}_{\text{energy flux through the boundary}} \xrightarrow{V\to 0}$$

- Momentum conservation $\partial_t T^{0i} = -\partial_x T^{xi} - \partial_y T^{yi} - \partial_z T^{zi}$

$\Longrightarrow$ Energy-momentum conservation:

$$\boxed{\partial_\mu T^{\mu\nu} = 0} \qquad (1.1)$$

- This is the most fundamental equation in fluid dynamics
    - Requires only continuosly distributed energy and momentum
- Conserved charges: Charge 4-current N_q^μ

  N_q^0 : charge density

  N_q^i : charge flux in i-diretion

- Charge conservation $\partial_t \int N_q^0 \, dV = -\oint N_q^i \, dA_i \overset{V\to 0}{\Longrightarrow} \partial_t N_q^0 = -\partial_i N_q^i$

$$\boxed{\partial_\mu N_q^\mu = 0} \qquad (1.2)$$

- Angular momentum tensor : $\mathcal{M}^{\alpha\beta\mu} = x^\alpha T^{\beta\mu} - x^\beta T^{\alpha\mu}$
  - angular momentum conservation

$$0 = \partial_\mu \mathcal{M}^{\alpha\beta\mu} = \underbrace{(\partial_\mu x^\alpha)}_{=\delta^\alpha_\mu} T^{\beta\mu} - \underbrace{(\partial_\mu x^\beta)}_{=\delta^\beta_\mu} T^{\alpha\mu} + x^\alpha \underbrace{\partial_\mu T^{\beta\mu}}_{=0} - x^\beta \underbrace{\partial_\mu T^{\alpha\mu}}_{=0}$$

$$= T^{\beta\alpha} - T^{\alpha\beta} \implies T^{\alpha\beta} = T^{\beta\alpha} \quad (T^{\mu\nu} \text{ symmetric})$$

- The basic quantities and equations of motion are now

$$\partial_\mu T^{\mu\nu} = 0 \qquad \partial_\mu N^\mu_q = 0$$

- Energy-momentum conseVation for continuous matter
- This defines fluid dynamics, but not yet very useful :
  - 5 equations but 14 unknowns ($T^{\mu\nu}$ - 10 (independent) components
    N^μ - 4 components)

- Continuous matter : motion described by fluid 4-velocity u^μ

- Position of fluid element $x^\mu(\tau)$, τ is a proper time (in comoving frame)

$$u^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dt}{d\tau}, \frac{d\vec{x}}{d\tau}\right) = \gamma(1, \vec{v}) \quad , \text{ where } \vec{v} = \frac{d\vec{x}}{dt} \quad , \gamma = \frac{1}{\sqrt{1-\vec{v}^2}} = \frac{dt}{d\tau}$$

- u^μ normalized : $u_\mu u^\mu = \gamma^2(1-\vec{v}^2) = 1$

- $u^\mu = (1, \vec{0})$ defines Local Rest Frame (LRF), where the fluid element is at rest.

- Using u^μ we can define

  - co-moving time derivative (material derivative) of quantity A

$$\overset{\circ}{A} = u^\mu \partial_\mu A \overset{LRF \,:\, u^\mu = (1,\vec{0})}{=} \underset{\substack{\uparrow \\ \text{now time in LRF}}}{\partial_t} A = \frac{d}{d\tau} A \qquad \text{(note } u^\mu\partial_\mu \text{ is scalar} \to \text{same in any frame)} \quad (1.3)$$

|| also $\underbrace{u^\mu \partial_\mu x^\nu}_{=\delta_\mu^\nu} = u^\nu$ (as it should)

note : $u^\mu \partial_\mu A = (\gamma \partial_t + \gamma \vec{v} \cdot \nabla) A$

|| non-relativistic :

$$\frac{dA}{d\tau} = (\partial_t + \vec{v}\cdot\nabla) A$$

- energy and charge density in LRF

$$u_\mu u_\nu T^{\mu\nu} = T^{00}_{LRF} = e \tag{1.4}$$

$$u_\mu N^\mu_q = N^0_{q,\,LRF} = n_q \tag{1.5}$$

- In fluid dynamics important distinction between convection and diffusion
  - Convection: energy/charge/momentum transport with flow u^μ
  - Diffusion: —— " —— without/orthogonal to flow
- Decomposition w.r.t. u^μ

$$N^\mu_q = N^\alpha_q \delta^\mu_\alpha = N^\alpha_q \left(u_\alpha u^\mu + \underbrace{\delta^\mu_\alpha - u_\alpha u^\mu}_{\equiv \Delta^\mu_\alpha} \right) = \underbrace{n_q u^\mu}_{\text{convection}} + \underbrace{V^\mu_q}_{\text{diffusion}} \quad V^\mu_q = \Delta^\mu_\nu N^\nu_q$$

  - $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is projection operator (orthogonal to u^μ)

$$\| \ \Delta^{\mu\nu} u_\mu = \Delta^{\mu\nu} u_\nu = 0 \qquad \& \qquad \Delta^\mu_\alpha \Delta^\alpha_\nu = \Delta^\mu_\nu$$

- $T^{\mu\nu}$ similarly

$$T^{\mu\nu} = T^{\alpha\beta}\,\delta^\mu_\alpha \delta^\nu_\beta = T^{\alpha\beta}\left(\delta^\mu_\alpha - u^\mu u_\alpha + u^\mu u_\alpha\right)\left(\delta^\nu_\beta - u^\nu u_\beta + u^\nu u_\beta\right)$$

$$= T^{\alpha\beta}\Delta^\mu_\alpha \Delta^\nu_\beta + T^{\alpha\beta}\Delta^\mu_\alpha u^\nu u_\beta + T^{\alpha\beta}\Delta^\nu_\beta u^\mu u_\alpha + \underbrace{T^{\alpha\beta}u_\alpha u_\beta}_{=e} u^\mu u^\nu$$

Define $W^\mu = T^{\alpha\beta}\Delta^\mu_\alpha u_\beta$

$$= e u^\mu u^\nu + W^\mu u^\nu + W^\nu u^\mu + T^{\alpha\beta}\underbrace{\Delta^\mu_\alpha \Delta^\nu_\beta}$$

symmetrize & separate trace

0, because $T^{\mu\nu}$ symmetric

$$\Delta^\mu_\alpha \Delta^\nu_\beta = \underbrace{\frac{1}{2}\left(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right)}_{=\Delta^{\mu\nu}_{\alpha\beta}} + \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta} + \text{antisymmetric}$$

$= \Delta^{\mu\nu}_{\alpha\beta}$ projection operator
symmetric, orthogonal to u^μ
& traceless

$$\Rightarrow \quad T^{\mu\nu} = e u^\mu u^\nu - P\Delta^{\mu\nu} + 2W^{(\mu}u^{\nu)} + \pi^{\mu\nu}$$

$$\left.\begin{aligned}
e &= T^{\mu\nu}u_\mu u_\nu && \text{(LRF energy density)} \\
P &= -\tfrac{1}{3}T^{\alpha\beta}\Delta_{\alpha\beta} && \text{(Isotropic pressure)} \\
W^\mu &= T^{\alpha\beta}\Delta^\mu_\alpha u_\beta && \text{(Energy diffusion current)} \\
\pi^{\mu\nu} &= \Delta^{\mu\nu}_{\alpha\beta}T^{\alpha\beta} \equiv T^{\langle\mu\nu\rangle} && \text{(Shear-stress tensor "momentum diffusion")}
\end{aligned}\right\} \quad (1.6)$$

$$N^\mu_q = n_q u^\mu + V^\mu_q$$

$$\left.\begin{aligned}
n_q &= N^\mu_q u_\mu && \text{(LRF charge density)} \\
V^\mu_q &= \Delta^\mu{}_\nu N^\nu_q && \text{(Charge diffusion current)}
\end{aligned}\right\} \quad (1.7)$$

- These are now the basic fluid dynamical quantities
- So far no help in closing the equations of motion (we have only introduced 3 new components of u^μ)

- A second ingredient (besides energy, momentum and charge conservation) is an existence of a local equilibrium state,

- In equilibrium we can relate temperature T, and thermodynamical pressure p_{eq} to e_{eq} and n_{eq} through equation of state (EoS)

$$p_{eq} = p_{eq}(e_{eq}, n_{eq}) \qquad T = T(e_{eq}, n_{eq}) \tag{1.8}$$

- In equilibrium : matter is locally isotropic (in LRF) $\Rightarrow$

$$T^{\mu\nu}_{eq} = e_{eq} u^\mu u^\nu - p_{eq} \Delta^{\mu\nu} \qquad N^\mu_{q,eq} = n_q u^\mu \tag{1.9}$$

- We can then split $T^{\mu\nu}$ into equilibrium and viscous parts

$$\begin{aligned} T^{\mu\nu} &= (e_{eq} + \delta e) u^\mu u^\nu - (p_{eq} + \Pi)\Delta^{\mu\nu} + 2 W^{(\mu} u^{\nu)} + \pi^{\mu\nu} \\ N^\mu &= (n_{eq} + \delta n) u^\mu + V^\mu_q \end{aligned} \tag{1.10}$$

here $\delta e = e - e_{eq}$ and $\delta n = n - n_{eq}$ are differences between actual densities and densities of an equilibrium state.

$\Pi = P - p_{eq}$ is difference btw. total isotropic pressure and eq. pressure (bulk viscous pressure)

- For a given state $T^{\mu\nu}$, equilibrium state is not unique, but need to be defined.

- A usual choice is the equilibrium state for which

$$e_{eq} = e \qquad \text{and} \qquad n_{eq} = n \qquad (\text{i.e. } \delta e = \delta n = 0) \qquad (1.11)$$

- These are often referred as (Landau) matching conditions.

- In equilibrium $e_{eq} = e_{eq}(T, \mu)$ $\qquad n_{eq} = n_{eq}(T, \mu)$

$\Rightarrow$ matching conditions can be thought as a definition of temperature and chemical potential for a non-equilibrium state.

$$\Rightarrow \quad T^{\mu\nu} = e u^\mu u^\nu - (P_{eq} + \Pi)\Delta^{\mu\nu} + 2W^{(\mu}u^{\nu)} + \pi^{\mu\nu}$$

$$N_q^\mu = n_q u^\mu + V_q^\mu \qquad (1.12)$$

- Dissipative quantities are now Π, W^μ, $\pi^{\mu\nu}$ and V_q^μ

Note: only definition of bulk viscous pressure Π need explicit reference to equilibrium – other dissipative quantities can be obtained by projecting $T^{\mu\nu}$ and N_q^μ with u^μ.

- Similarly as equilibrium state, fluid velocity is not unique (only in equilibrium), but needs to be defined.

- In order to define u^μ we need to decide what "flows"

- Two common choices : net-charge flow (Eckart frame), or total energy flow (Landau frame)

  - Eckart frame : $u^\mu = \dfrac{N_q^\mu}{\sqrt{N_\mu^q N_q^\mu}} \Rightarrow \Delta^{\mu\nu} = g^{\mu\nu} - \dfrac{N_q^\mu N_q^\nu}{N_q \cdot N_q}$

$$\Rightarrow V_q^\mu = N_q^\nu \Delta^\mu{}_\nu = N_q^\mu - N_q^\mu \frac{N_q \cdot N_q}{N_q \cdot N_q} = \underline{\underline{0}}$$

$\Rightarrow$ Charge diffusion vanishes (no flow of charge $\perp u^\mu$)

  - Landau frame : $T^{\mu\nu} u_\nu = e u^\mu$

In general : $T^{\mu\nu} u_\nu = e u^\mu + W^\mu$

$\Rightarrow$ In Landau frame energy diffusion vanishes

$$\underline{\underline{W^\mu = 0}}$$

Let's use this from now on. Eckart frame is sometimes ill defined, e.g. when $n_q = 0$.

- Energy-momentum tensor and charge 4-current are now expressed as

$$T^{\mu\nu} = e u^\mu u^\nu - (p_{eq}(e, n_q) - \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

$$N_q^\mu = n_q u^\mu + V_q^\mu$$

- Equilibrium state defined through matching $e_{eq}(T, \mu) = e \qquad n_{eq}(T, \mu) = n_q$

- Fluid 4-velocity defined through $T^{\mu\nu} u_\nu = e u^\mu$ (Landau frame)

- We can now cast the conservation laws into a more understandable form

$$0 = u_\nu \partial_\mu T^{\mu\nu} = u_\nu \partial_\mu \left[e u^\mu u^\nu - (p_{eq} + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} \right]$$

$\Vert$ use

$$u^\mu \partial_\mu e = \dot{e}$$
$$u^\mu \partial_\nu u_\mu = 0 \qquad (u^\mu u_\mu = 1 \Rightarrow \partial_\nu (u^\mu u_\mu) = 0 = 2u^\mu \partial_\nu u_\mu)$$
$$\partial_\mu g^{\mu\nu} = 0$$
$$u_\nu \partial_\mu \pi^{\mu\nu} = -\pi^{\mu\nu} \partial_\mu u_\nu$$

$$\Rightarrow \dot{e} = -(e + p_{eq} + \Pi)\underline{\underline{\partial_\mu u^\mu}} + \pi^{\mu\nu} \underline{\underline{\partial_\mu u_\nu}}$$

- Evolution of energy density driven by gradients of u^μ
- Decompose $\partial_\mu u_\nu$
  - Define 3-gradient : $\nabla_\mu = \Delta_\mu{}^\nu \partial_\nu \overset{\text{LRF}}{=} (0, \vec{\nabla}) \quad \| \quad u^\mu \nabla_\mu = 0$

$$\partial_\mu = \delta_\mu^\alpha \partial_\alpha = (u^\alpha u_\mu + \underbrace{\delta_\mu^\alpha - u^\alpha u_\mu}_{=\Delta_\mu^\alpha})\partial_\alpha = u_\mu \underbrace{u^\alpha \partial_\alpha}_{\frac{d}{d\tau}} + \nabla_\mu = u_\mu \frac{d}{d\tau} + \nabla_\mu$$

$$\Rightarrow \partial_\mu u_\nu = u_\mu \frac{d}{d\tau} u_\nu + \underbrace{\nabla_\mu u_\nu}$$

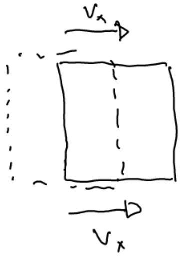
$$= \underbrace{\frac{1}{2}(\nabla_\mu u_\nu + \nabla_\nu u_\mu)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(\nabla_\mu u_\nu - \nabla_\nu u_\mu)}_{\text{antisymmetric}}$$

$$= u_\mu \dot{u}_\nu + \underbrace{\frac{1}{2}\left(\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}\Delta_{\mu\nu} \nabla_\alpha u^\alpha\right)}_{=\sigma_{\mu\nu}} + \frac{1}{3}\Delta_{\mu\nu} \underbrace{\nabla_\alpha u^\alpha}_{=\theta} + \underbrace{\frac{1}{2}(\nabla_\mu u_\nu - \nabla_\nu u_\mu)}_{=\omega_{\mu\nu}}$$

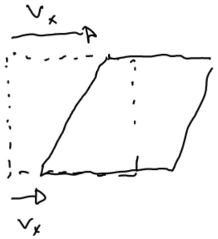
add and remove trace

$$\Rightarrow \partial_\mu u_\nu = u_\mu \dot{u}_\nu + \sigma_{\mu\nu} + \frac{1}{3}\Delta_{\mu\nu}\theta + \omega_{\mu\nu}$$

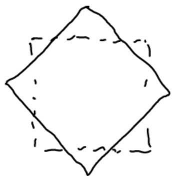
- Different parts here correspond different deformations of fluid element



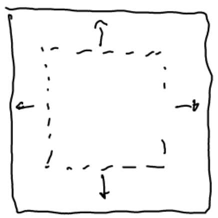
$u_\mu \dot{u}_\nu$ = acceleration



$\sigma_{\mu\nu} = \Delta^{\alpha\beta}_{\mu\nu} \partial_\alpha u_\beta$ = shear deformation (do not change volume, that is why we removed trace)



$\omega_{\mu\nu}$ = rotations (vorticity tensor)



$\Delta_{\mu\nu} \theta$ = expansion $\theta = \nabla_\mu u^\mu = \partial_\mu u^\mu$ = volume expansion rate

- Different type of deformations assosiated with different dissipative processes

- Back to energy equation

$$\dot{e} = -(e + p_{eq} + \Pi)\underbrace{\partial_\mu u^\mu}_{=\Theta} + \Pi^{\mu\nu}\underbrace{\partial_\mu u_\nu}$$

$$= \underbrace{u_\mu \dot{u}_\nu}_{0} + \sigma_{\mu\nu} + \underbrace{\omega_{\mu\nu}}_{0} + \frac{1}{3}\Delta_{\mu\nu}\Theta$$

$(\Pi^{\mu\nu}u_\mu = 0)$ (antisymmetric $\wedge$ symmetric) $\rightarrow 0 \quad \Delta_{\mu\nu}\Pi^{\mu\nu} = \Pi^\mu{}_\mu = 0$

$$\Rightarrow \quad \dot{e} = -e\Theta - (p_{eq} + \Pi)\Theta + \underbrace{\Pi^{\mu\nu}\sigma_{\mu\nu}}$$

expansion ($-e\Theta$), $p\,dV$-work ($-(p_{eq}+\Pi)\Theta$), $\rightarrow$ dissipative heating (motion $\rightarrow$ heat)

- Second equation of motion similarly

$$\Delta^\mu_\alpha \partial_\nu T^{\nu\alpha} = 0$$

$$\Rightarrow (e + p_{eq} + \Pi)\dot{u}^\mu = \underbrace{\nabla^\mu(p_{eq} + \Pi)}_{\text{"force"}} - \underbrace{\nabla_\alpha \Pi^{\mu\alpha}}_{\text{"viscosity"}}$$

(Relativistic Newton II for continuous matter)

- Charge conservation

$$\partial_\mu N^\mu_q = \partial_\mu \left(n_q u^\mu + V^\mu_q \right) = \dot{n}_q + n_q \theta + \partial_\mu V^\mu_q = 0$$

$$\Rightarrow \dot{n}_q = \underbrace{-n_q \theta}_{\text{expansion}} - \underbrace{\partial_\mu V^\mu_q}_{\hookrightarrow \text{diffusion}}$$

- We have now written the conservation laws in terms of our new fluid variables

$$\dot{e} = -(e + P_{eq} + \Pi)\theta + \pi^{\mu\nu}\sigma_{\mu\nu}$$

$$(e + P_{eq} + \Pi)\dot{u}^\mu = \nabla^\mu (P_{eq} + \Pi) + \nabla_\alpha \pi^{\mu\alpha}$$

$$\dot{n}_q = -n_q \theta - \partial_\mu V^\mu_q$$

- Doesn't solve the problem that we have 14 unknowns and only 5 equations

  - Need 9 additional equations to close the system :

$$\underset{(5)}{\pi^{\mu\nu} = \pi^{\mu\nu}(e, n_q, \nabla u, \nabla e, \nabla n_q)}, \quad \underset{(3)}{V^\mu_q = V^\mu_q (e, n, \nabla u, \nabla e, \nabla n)} \quad \underset{(1)}{\theta = \theta(e, n_q, \nabla u, \nabla n_q, \nabla e)}$$

- Usually we think fluid dynamics in situation, where densities, and velocity change slowly in space and time

  $\Rightarrow$ Include only 1st-order gradients

$$
\begin{cases}
\pi^{\mu\nu} = 2\eta\,\sigma^{\mu\nu} \\
V_q^{\mu} = \kappa\,\nabla^{\mu}\left(\frac{\mu}{T}\right) \\
\Pi = -\zeta\,\theta
\end{cases}
$$

- η, κ, and ζ positive : material properties (like EoS)

  $\eta(T,\mu)$ shear viscosity
  $\kappa(T,\mu)$ diffusion constant
  $\zeta(T,\mu)$ bulk viscosity

- This is relativistic generalization of Newtonian fluids

  (usually referred as relativistic Navier-Stokes theory)

- $\frac{\pi^{\mu\nu}}{p_{eq}} = \underbrace{\frac{2\eta}{p_{eq}}}_{\text{dimension fm}} \underbrace{\sigma^{\mu\nu}}_{\nabla u \text{ dimension fm}^{-1}}$ (macroscopic scale over which velocity varies

  $\frac{2\eta}{p_{eq}} \sim \lambda_{mfp}$ (microscopic scale)

  $\Rightarrow \frac{2\eta}{p_{eq}} \sigma^{\mu\nu} \sim O(Kn)$

- Relativistic Navier-Stokes theory is 1st order in Knudsen number Kn

  $Kn = \frac{l_{micr}}{L_{macr}}$

- Kn quantifies degree of separation between microscopic and macroscopic scales

- Fluid dynamics when $Kn \lesssim 1$

- As presented here, relativistic Navier-Stokes theory is not a good relativistic theory:
  - signal propagation speed can exceed speed of light $\rightarrow$ acausal theory

- Problem is that is NS theory the microscopic state of matter (e.g. $\pi^{\mu\nu}$) reacts immediately to changes in external conditions e.g ∇u or $\sigma^{\mu\nu}$.
  In reality this takes time τ_{mfc} between particle collisions.

- We can incorporate τ_{mfc} into fluid dynamics, and resolve it explicitly
  $\longrightarrow$ Transient fluid dynamics (e.g. Israel-Stewart theory)

- $\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} \implies \dot{\pi}^{\langle\mu\nu\rangle} = \frac{1}{\tau_\pi}\left(2\eta\sigma^{\mu\nu} - \pi^{\mu\nu}\right)$ + higher order terms

  + similarly for $\overset{\circ}{V}{}^{\mu}_{q}$ and Π

- Next our goal is to derive this type of fluid dynamics from kinetic theory of relativistic gases.

2. Elements of kinetic theory

- The basic quantity in kinetic theory is a single-particle distributio function

$$f(x,k) \equiv f_k$$

- This is probability density for observing particle at spacetime point $[x, x+dx]$ with momentum $[k, k+dk]$
- Macroscopic densities and fluxes can be obtained by integrating over k
- Particle density $N^0 = \int \underbrace{\frac{d^3\vec{k}}{(2\pi)^3}}_{\text{density of states}} f_k$
- Particle flux $N^i = \int \frac{d^3\vec{k}}{(2\pi)^3} \underbrace{\frac{k^i}{k^0}}_{\text{particle velocity}} f_k \quad , \; k^0 = \sqrt{\vec{k}^2 + m^2}$

combine both:

$\Rightarrow$ Particle 4-current $$N^\mu = \int \underbrace{\frac{d^3\vec{k}}{(2\pi)^3 k^0}}_{\text{scalar}} \underbrace{k^\mu}_{\text{4-vector}} \underbrace{f_k}_{\text{scalar}} \qquad (2.1)$$

- Energy density $\quad T^{00} = \int \frac{d^3\vec{k}}{(2\pi)^3} k^0 f_k$
- Energy flux $\quad T^{i0} = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{k^i}{k^0} k^0 f_k$
- Momentum density $\quad T^{0i} = \int \frac{d^3\vec{k}}{(2\pi)^3} k^i f_k$
- j- momentum flux in i-direction $\quad T^{ij} = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{k^i}{k^0} k^j f_k$

combine $\Longrightarrow$ $$T^{\mu\nu} = \int \frac{d^3\vec{k}}{(2\pi)^3 k^0} k^\mu k^\nu f_k \qquad (2.2)$$

notation $$dK = \frac{d^3\vec{k}}{(2\pi)^3 k^0} \qquad (2.3)$$

- We have now our basic fluid quantities expressed as integrals over f_k

$$T^{\mu\nu} = \int dK\, k^\mu k^\nu f_k \qquad N^\mu = \int dK\, k^\mu f_k \qquad (2.4)$$

- We can now decompose $T^{\mu\nu}$ and N^μ as before, so that definitions (1.6) remain the same (still use Landau velocity)

$$T^{\mu\nu} = e u^\mu u^\nu - (p_{eq} + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} \qquad N^\mu = n u^\mu + V^\mu$$

$$e = T^{\mu\nu} u_\mu u_\nu = \int dK \, E_k^2 f_k \quad , \text{ where } \quad E_k = k^\mu u_\mu \overset{\text{LRF}}{=} k^0_{\text{LRF}} \quad \text{(particle energy in LRF)}$$

$$p_{eq} + \Pi = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu} = -\frac{1}{3}\int dK \, (\Delta_{\mu\nu} k^\mu k^\nu) f_k$$

$$\pi^{\mu\nu} = T^{\langle\mu\nu\rangle} = \int dK \, k^{\langle\mu} k^{\nu\rangle} f_k \qquad (2.5)$$

$$n = N^\mu u_\mu = \int dK \, E_k f_k$$

$$V^\mu = N^\alpha \Delta^\mu_\alpha = \int dK \, k^{\langle\mu\rangle} f_k \quad , \text{ where } \quad k^{\langle\mu\rangle} = \Delta^\mu_\alpha k^\alpha \qquad k^{\langle\mu\rangle} u_\mu = 0$$

- Equilibrium distribution f_{eq} is given by Bose-Einstein or Fermi-Dirac distr.

$$f_{eq}(T,\mu) = \left[e^{(E_k - \mu)/T} + a \right]^{-1} \qquad a = \begin{cases} 1 \, , & \text{fermions} \\ -1 \, , & \text{bosons} \\ 0 \, , & \text{classical particles} \end{cases} \qquad (2.6)$$

- As before, we wish to include equilibrium state explicitly, and write

$$f_k = f_{eq} + \delta f \qquad (2.7)$$

- δf is now deviation from equilibrium distribution, in equilibrium $\delta f = 0$

- For a general f_k equilibrium state is not unique: invoke matching conditions

$$\left.\begin{aligned} e &= \int dK\, E_k^2 f_k = e_{eq} = \int dK\, E_k^2 f_{eq} \\ n &= \int dK\, E_k f_k = n_{eq} = \int dK\, E_k f_{eq} \end{aligned}\right\} \Rightarrow \begin{aligned} \delta e &= \int dK\, E_k^2 \delta f = 0 \\ \delta n &= \int dK\, E_k \delta f = 0 \end{aligned} \qquad (2.8)$$

- δf does not contribute energy nor particle density

- These conditions define T and μ for a general off-equilibrium state f_k

- In equilibrium:

integrand: only tensors u^μ and $g^{\mu\nu}$ (k^μ integrated)

$$\left.\begin{aligned} T^{\mu\nu}_{eq} &= \int dK\, k^\mu k^\nu f_{eq}(E_k = u\cdot k) = A u^\mu u^\nu + B g^{\mu\nu} = \underline{\underline{e_{eq} u^\mu u^\nu - P_{eq}\Delta^{\mu\nu}}} \\ N^\mu_{eq} &= \int dK\, k^\mu f_{eq} = \underline{\underline{n u^\mu}} \end{aligned}\right) \qquad (2.9)$$

$\Delta^{\mu\nu}_{\alpha\beta} u^\alpha = 0$ $\qquad$ $\Delta^{\mu\nu\alpha}{}_{\alpha} = 0$

$$\left.\begin{aligned} \Rightarrow \quad \pi^{\mu\nu}_{eq} &= \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta}_{eq} = \int dK\, k^{\langle\mu} k^{\nu\rangle} f_{eq} = \Delta^{\mu\nu}_{\alpha\beta}\left(e_{eq} u^\alpha u^\beta - P_{eq}\Delta^{\alpha\beta}\right) = \underline{\underline{0}} \\ V^\mu_{eq} &= \Delta^\mu_\alpha N^\alpha_{eq} = \int dK\, k^{\langle\mu\rangle} f_{eq} = \Delta^\mu_\alpha (n u^\alpha) = \underline{\underline{0}} \end{aligned}\right) \qquad (2.10)$$

- These are not restricted to equilibrium, but any $F_k(E_k)$ that is only function of E_k give

$$\underbrace{\int dK\, k^{\langle\mu}k^{\nu\rangle} F_k = \int dK\, k^{\langle\mu\rangle} F_k}_{\text{directional information}} = 0 \qquad (2.11)$$

(F_k isotropic in LRF)

- Our expression of f_k is now

$$f_k = \underbrace{f_{eq}}_{\substack{\text{depends on } u^\mu,\, T,\, \mu}} + \delta f \qquad (2.12)$$

depends on u^μ, T, μ → defined by velocity frame and matching conditions

- Obviously f_k has still infinitely many degrees of freedom, and 5 constraints from the conservation laws $\partial_\mu T^{\mu\nu} = 0$ and $\partial_\mu N^\mu = 0$ do not constrain it much.

- Full dynamics of f_k is given by the relativistic Boltzmann equation

$$k^\mu \partial_\mu f_k = \underbrace{C[f_k]}_{\substack{\text{collision}\\ \text{integral}}} \qquad (2.13)$$

- Collision integral for binary collision $k + k' \longrightarrow p + p'$

$$C[f_k] = \frac{1}{2} \int dK' dP dP' \, W_{kk' \to pp'} \left(f_p f_{p'} \tilde{f}_k \tilde{f}_{k'} - f_k f_{k'} \tilde{f}_p \tilde{f}_{p'} \right) \qquad (2.14)$$

$W_{kk' \to pp'}$ transition rate , $\underbrace{\tilde{f}_k = 1 - a f_k}_{\text{quantum statistics}}$

- Question now is: can we reduce the complicated dynamics given by the Boltzmann equation, and write the dynamics in terms of just few macroscopic quantities like $T^{\mu\nu}$ and N^μ

i.e. can we write fluid dynamical limit of Boltzmann equation

- For this purpose we first write the Boltzmann equation entirely in terms of macroscopic quantities that are moments of f_k.

e.g. $\pi^{\mu\nu} = \int dK \, k^{\langle\mu} k^{\nu\rangle} f_k$, but also $\underbrace{\rho_n^{\mu\nu} = \int dK E_k^n \, k^{\langle\mu} k^{\nu\rangle} f_k}_{}$

There are infinitely many such quantities $\Longleftrightarrow$ infinitely many d.o.f. in f_k

3. Expansion of f_k

- $$f_k = f_{eq} + \delta f = f_{eq}\left(1 + \tilde{f}_{eq}\phi\right) \qquad (3.1)$$
  $\tilde{f}_{eq} \to 1 - af_{eq}$

- Expand f_k around equilibrium (in momentum space)

- Expansion basis $1, k^\mu, k^\mu k^\nu, k^\mu k^\nu k^\lambda, \ldots$ (like Israel and Stewart)

$$\Longrightarrow \phi = \varepsilon + \varepsilon^\mu k_\mu + \varepsilon^{\mu\nu} k_\mu k_\nu + \varepsilon^{\mu\nu\lambda} k_\mu k_\nu k_\lambda + \cdots \qquad (3.2)$$

  - ε's are the expansion coefficients

- One way to reduce the degrees of freedom is to directly truncate the expansion.

- Israel & Stewart 14-moment approximation

$$\phi = \varepsilon + \dot{\varepsilon}^\mu k_\mu + \varepsilon^{\mu\nu} k_\mu k_\nu \qquad (3.3)$$

- The coefficients can be determined by requiring

$$T^{\mu\nu} = \int dK\, k^\mu k^\nu f_k = \int dK\, k^\mu k^\nu f_{eq} + \int dK\, k^\mu k^\nu f_{eq} \tilde{f}_{eq} \left(\varepsilon + \varepsilon^\mu k_\mu + \varepsilon^{\mu\nu} k_\mu k_\nu \right)$$

$$N^\mu = \int dK\, k^\mu f_k = \int dK\, k^\mu f_{eq} + \int dK\, k^\mu f_{eq} \tilde{f}_{eq} \left(\varepsilon + \varepsilon^\mu k_\mu + \varepsilon^{\mu\nu} k_\mu k_\nu \right)$$

$\Longrightarrow$ express ε's in terms of $T^{\mu\nu}$ and N^μ

Example:

$$\underline{\underline{\pi^{\mu\nu}}} = \Delta^{\mu\nu}_{\alpha\beta} \int dK\, k^\alpha k^\beta f_k = \underbrace{\Delta^{\mu\nu}_{\alpha\beta} \int dK\, k^\alpha k^\beta f_{eq}}_{\to 0} + \Delta^{\mu\nu}_{\alpha\beta} \int dK\, f_{eq} \tilde{f}_{eq}\, k^\alpha k^\beta \left(\underbrace{\varepsilon}_{\to 0} + \varepsilon^\gamma k_\gamma + \varepsilon^{\gamma\delta} k_\gamma k_\delta \right)$$

$$= \underline{\Delta^{\mu\nu}_{\alpha\beta}} \underbrace{\int dK\, f_{eq} \tilde{f}_{eq}\, k^\alpha k^\beta k^\gamma \varepsilon_\gamma}_{= \varepsilon_\gamma \left[A \underbrace{u^\alpha u^\beta u^\gamma}_{\to 0,\ \perp} + B \underbrace{u^{(\alpha} \Delta^{\beta\gamma)}}_{\to 0,\ \perp \,\&\, \text{trace}} \right]} + \Delta^{\mu\nu}_{\alpha\beta} \underbrace{\int dK\, f_{eq} \tilde{f}_{eq}\, k^\alpha k^\beta k^\gamma k^\delta \varepsilon_{\gamma\delta}}_{= \varepsilon_{\gamma\delta} \left[A' \underbrace{u^\alpha u^\beta u^\gamma u^\delta}_{\to 0} + B' \underbrace{u^{(\alpha} u^\beta \Delta^{\gamma\delta)}}_{\to 0} + C' \Delta^{(\alpha\beta} \Delta^{\gamma\delta)} \right]}$$

$$= \Delta^{\mu\nu}_{\alpha\beta}\, \varepsilon_{\gamma\delta}\, C' \Delta^{(\alpha\beta} \Delta^{\gamma\delta)} = \varepsilon_{\gamma\delta}\, C' \Delta^{\mu\nu}_{\alpha\beta} \frac{1}{3} \left(\underbrace{\Delta^{\alpha\beta} \Delta^{\gamma\delta}}_{\to 0} + \Delta^{\alpha\gamma} \Delta^{\beta\delta} + \Delta^{\alpha\delta} \Delta^{\beta\gamma} \right) = \frac{2}{3} \varepsilon_{\gamma\delta}\, C' \Delta^{\mu\nu\gamma\delta}$$

$$= \frac{2}{3} C' \underline{\underline{\varepsilon^{\langle\mu\nu\rangle}}}$$

- Coefficient C' can be computed from

$$J^{\alpha\beta\gamma\delta} = \int dK\, f_{eq} \tilde{f}_{eq}\, k^\alpha k^\beta k^\gamma k^\delta = A' u^\alpha u^\beta u^\gamma u^\delta + B' u^{(\alpha} u^\beta \Delta^{\gamma\delta)} + C' \Delta^{(\alpha\beta} \Delta^{\gamma\delta)}$$

by taking projection $\Delta_{\alpha\beta}\Delta_{\gamma\delta} J^{\alpha\beta\gamma\delta} \rightarrow C' = \frac{1}{5}\int dK \left(\Delta_{\alpha\beta} k^\alpha k^\beta\right)^2 f_{eq}\tilde{f}_{eq}$

- This is usually written as

$$C' = 3 J_{42}$$

- Definition

$$J_{nq} = \frac{1}{(2q+1)!!}\int dK\, E_k^{n-2q} \left(-\Delta^{\alpha\beta} k_\alpha k_\beta\right)^q f_{eq}\tilde{f}_{eq} \qquad (3.4)$$

$\hookrightarrow$ if $\tilde{f}_{eq} = 1$ $J_{nq} \rightarrow I_{nq}$

$$\Rightarrow \pi^{\mu\nu} = 2 J_{42}\, \varepsilon^{\langle\mu\nu\rangle}$$

- If bulk viscosity and diffusion can be neglected $\longrightarrow$ often used approximation in heavy-ion phenomenology

$$\delta f = \frac{f_{eq}}{2 J_{42}} \pi^{\mu\nu} k_{\langle\mu} k_{\nu\rangle} \qquad (3.5)$$

- We introduced 14-moment approximation

$$\phi = \varepsilon + \varepsilon^{\mu} k_{\mu} + \varepsilon^{\mu\nu} k_{\mu} k_{\nu}$$

- By matching to $T^{\mu\nu}$ and N^{μ} we can write ε's in terms of dissipative quantities Π, V^{μ}, $\pi^{\mu\nu}$

- So far no dynamics, but we can see that time-evolution of $T^{\mu\nu}$ and N^{μ} give also time-evolution of f_k (in 14 moment approximation)

- We can generalize 14-moment approximation to any number of moments.

- For this purpose we modify the expansion basis somewhat

$$1,\ k^{\mu},\ k^{\mu}k^{\nu},\ k^{\mu}k^{\nu}k^{\lambda}, \dots \longrightarrow 1,\ k^{\langle\mu\rangle},\ k^{\langle\mu}k^{\nu\rangle},\ k^{\langle\mu}k^{\nu}k^{\lambda\rangle}, \dots \qquad (3.6)$$

where tensors $k^{\langle\mu_1} \cdots k^{\mu_m\rangle} = \Delta^{\mu_1 \cdots \mu_m \nu_1 \cdots \nu_m} k_{\nu_1} \cdots k_{\nu_m}$ are symmetric, orthogonal, and traceless that satisfy orthogonality condition

$$\int dK F_k\, k^{\langle\mu_1} \cdots k^{\mu_m\rangle} k^{\langle\nu_1} \cdots k^{\nu_n\rangle} = \frac{m!\,\delta_{mn}}{(2m+1)!!} \Delta^{\mu_1 \cdots \mu_m \nu_1 \cdots \nu_m} \int dK F_k \left(\Delta^{\alpha\beta} k_{\alpha} k_{\beta}\right)^m \qquad (3.7)$$

where F_k is an arbitrary function of E_k (F_k isotropic in LRF)

- Using this basis we can write f_k as

$$f_k = f_{eq}\left[1 + \tilde{f}_{eq} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \mathcal{H}_n^{(\ell)} \rho_n^{\mu_1 \cdots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle}\right] \tag{3.8}$$

where $\mathcal{H}_n^{(\ell)}$ is polynomial in E_k

- Irreducible tensors $\rho_n^{\mu_1 \cdots \mu_\ell}$ are defined as

$$\rho_n^{\mu_1 \cdots \mu_\ell} = \int dK\, E_k^n\, k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \delta f \tag{3.9}$$

- These are independent of momentum $\vec{k}$ and some of them can be identified as dissipative fluid variables

$$\rho_0 = -\frac{3\Pi}{m^2} \qquad \rho_0^\mu = V^\mu \qquad \rho_1^\mu = W^\mu \overset{\text{Landau } u^\mu}{=} 0\ , \quad \rho_0^{\mu\nu} = \pi^{\mu\nu} \qquad \Big\| \quad \begin{array}{l} e,n\text{ - matching}: \\ \rho_1 = \rho_2 = 0 \end{array} \tag{3.10}$$

- Once we have truncated the expansion (3.8) for any number of terms, the polynomials $\mathcal{H}_n^{(\ell)} = \sum_{i=0} a_{ni}^{(\ell)} E_k^i$ can be determined by requiring consistency between (3.8) and (3.9).

- Alternatively it is possible to use orthogonal polynomials $\rho_n^{\mu_1 \cdots \mu_\ell} \rightarrow c_n^{\mu_1 \cdots \mu_\ell} = \int dK P_n^{(\ell)} k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \delta f$
  $\mathcal{H}_n^{(\ell)} \rightarrow P_n^{(\ell)}$, where $P_n^{(\ell)}$ is set of orthogonal polynomials (not used here)

- What we have now achieved is that we have written f_k in terms of (infinite) set of macroscopic quantities $\rho_n^{\mu_1 \cdots \mu_\ell}$

- Once we know dynamics of $\{\rho_n^{\mu_1 \cdots \mu_\ell}\} \Longrightarrow$ dynamics of f_k

- Remaining task is to write equations of motion for $\rho_n^{\mu_1 \cdots \mu_\ell}$

- Write co-moving derivative of $\rho_n^{\mu_1 \cdots \mu_\ell}$:

$$\Delta^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} \frac{d}{d\tau} \rho_n^{\nu_1 \cdots \nu_\ell} = \mathring{\rho}^{\langle \mu_1 \cdots \mu_\ell \rangle} = \Delta^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} \frac{d}{d\tau} \int dK\, E_k^n\, k^{\langle \nu_1} \cdots k^{\nu_n \rangle} \delta f_k \tag{3.11}$$

- Using decompositions $f_k = f_{eq} + \delta f$, $k^\mu = E_k u^\mu + k^{\langle \mu \rangle}$, and $\partial_\mu = u_\mu \frac{d}{d\tau} + \nabla_\mu$ we can write the Boltzmann equation as

$$\mathring{\delta f} = -\dot{f}_{eq} - E_k^{-1} k_\nu \nabla^\nu f_{eq} - E_k^{-1} k_\nu \nabla^\nu \delta f_k + E_k^{-1} C[f_k] \tag{3.12}$$

- Substituting this to (3.11) we obtain exact equations for $\mathring{\rho}_n^{\langle \mu_1 \cdots \mu_\ell \rangle}$

- This is somewhat long computation and here are only some steps as an example for rank-2 tensors.

$$\dot{\rho}_r^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta}\frac{d}{d\tau}\int dK E_k^r k^{\langle\alpha}k^{\beta\rangle}\delta f = \Delta^{\mu\nu}_{\alpha\beta}\int dK \frac{d}{d\tau}\left(E_k^r k^{\langle\alpha}k^{\beta\rangle}\right)\delta f + \Delta^{\mu\nu}_{\alpha\beta}\int dK E_k^r k^{\langle\alpha}k^{\beta\rangle}\delta\dot{f} \tag{3.13}$$

- substitute Boltzmann equation

$$= \Delta^{\mu\nu}_{\alpha\beta}\int dK \frac{d}{d\tau}\left(E_k^r k^{\langle\alpha}k^{\beta\rangle}\right)\delta f + \Delta^{\mu\nu}_{\alpha\beta}\int dK E_k^r k^{\langle\alpha}k^{\beta\rangle}\Big[\underbrace{-\dot{f}_{eq}}_{1^\circ} \underbrace{-E_k^{-1}k_\lambda\nabla^\lambda f_{eq}}_{2^\circ} - E_k^{-1}k_\lambda\nabla^\lambda\delta f_k \underbrace{+E_k^{-1}C[f]}_{3^\circ}\Big]$$

$1^\circ = \Delta^{\mu\nu}_{\alpha\beta}\int dK E_k^r k^\alpha k^\beta \dot{f}_{eq}$

$$\dot{f}_{eq} = \frac{d}{d\tau}\left[e^{\frac{E_k-\mu}{T}}+a\right]^{-1} = \frac{d}{d\tau}\left(\frac{E_k-\mu}{T}\right)\underbrace{e^{\frac{E_k-\mu}{T}}\left[e^{(E_k-\mu)/T}+a\right]^{-2}}$$

$E_k := k^\gamma u_\gamma$

$$= \left[\frac{\dot{u}_\gamma k^\gamma}{T} - \frac{E_k}{T^2}\dot{T} - \frac{\dot{\mu}}{T}\right] f_{eq}\tilde{f}_{eq}$$

$$e^x\left[e^x+a\right]^{-2} = \frac{e^x+a-a}{[e^x+a]^2} = [e^x+a]^{-1}\left[1-a[e^x+a]^{-1}\right] = f_{eq}\tilde{f}_{eq}$$

$$\Delta^{\mu\nu}_{\alpha\beta}\,\dot{u}_\gamma\frac{1}{T}\underbrace{\int dK E_k^r k^\gamma k^\alpha k^\beta f_{eq}\tilde{f}_{eq}}_{A' u^\gamma u^\alpha u^\beta \,\to\, 0 \;+\; B' u^{(\gamma}\Delta^{\alpha\beta)} \,\to\, 0} - \Delta^{\mu\nu}_{\alpha\beta}\frac{\dot{T}}{T^2}\underbrace{\int dK(\cdots)}_{\to 0 \text{ (similarly)}}$$

$$\Longrightarrow \Delta^{\mu\nu}_{\alpha\beta}\int dK E_k^r k^\alpha k^\beta \dot{f}_{eq} = 0 \tag{3.14}$$

2° : $\Delta^{\mu\nu}_{\alpha\beta} \int dK\, E_k^r\, k^{\langle\alpha} k^{\beta\rangle} \left[E_k^{-1} k_\lambda \nabla^\lambda f_{eq} \right]$ (3.15)

take this outside of integral

$$= \Delta^{\mu\nu}_{\alpha\beta} \nabla_\lambda \underbrace{\int dK\, E_k^{r-1} k^\alpha k^\beta k^\lambda f_{eq}}_{\text{equilibrium moments}} - (r-1) \Delta^{\mu\nu}_{\alpha\beta} \nabla_\lambda u_\rho \underbrace{\int dK\, E_k^{r-2} k^\alpha k^\beta k^\lambda k^\rho f_{eq}}_{\text{equilibrium moments}}$$

decompose

$$\int dK\, E_k^{r-1} k^\alpha k^\beta k^\gamma f_{eq} = I_{r+2,0}\, u^\alpha u^\beta u^\gamma - 3 I_{r+2,1}\, u^{(\alpha} \Delta^{\beta\gamma)}$$

$$\int dK\, E_k^{r-2} k^\alpha k^\beta k^\lambda k^\rho = I_{r+2,0}\, u^\alpha u^\beta u^\lambda u^\gamma - 6 I_{r+2,1}\, u^{(\alpha} u^\rho \Delta^{\beta\lambda)} + 3 I_{r+2,2}\, \Delta^{\alpha(\beta}\Delta^{\lambda\rho)}$$

- Only the last terms with $I_{r+2,1}$ and $I_{r+2,2}$ will survive $\Delta^{\mu\nu}_{\alpha\beta}$

$$\Rightarrow \Delta^{\mu\nu}_{\alpha\beta} \int dK\, E_k^r\, k^{\langle\alpha} k^{\beta\rangle} \left[E_k^{-1} k_\lambda \nabla^\lambda f_{eq} \right] = -2 \left[I_{r+2,1} + (r-1) I_{r+2,2} \right] \underbrace{\Delta^{\mu\nu}_{\alpha\beta} \nabla^\alpha u^\beta}_{= \sigma^{\mu\nu}}$$

3° $\Delta^{\mu\nu}_{\alpha\beta} \int dK\, E_k^{r-1} k^\alpha k^\beta C[f]$ (3.16)

$$= \Delta^{\mu\nu}_{\alpha\beta} \int dK\, E_k^{r-1} k^\alpha k^\beta \frac{1}{2} \int dK' dP dP'\, W_{kk' \to pp'} \left(f_p f_{p'} \tilde{f}_k \tilde{f}_{k'} - f_k f_{k'} \tilde{f}_p \tilde{f}_{p'} \right)$$

- Linearizing the collision integral

$$\frac{1}{2}\int dK' dP dP' \, W_{kk'\to pp'} \left(f_p f_{p'} \tilde{f}_k \tilde{f}_{k'} - f_k f_{k'} \tilde{f}_p \tilde{f}_{p'} \right) \qquad (3.17)$$

- use $f_k = f_{eq}\left(1 + \tilde{f}_{eq}\phi_k\right)$ and keep only terms 1st order in ϕ

$$f_p f_{p'} = f_{eq,p} f_{eq,p'} \left(1 + \tilde{f}_{eq,p'}\phi_{p'} + \tilde{f}_{eq,p}\phi_p\right) + O(\phi^2)$$

$$\tilde{f}_p \tilde{f}_{p'} = \tilde{f}_{eq,p} \tilde{f}_{eq,p'} \left(1 - a f_{eq,p'}\phi_{p'} - a f_{eq,p}\phi_p\right) + O(\phi^2)$$

- Using further

$$\tilde{f}_{eq,p} = f_{eq,p} \exp\left[\frac{E_p - \mu}{T}\right] \qquad (3.18)$$

$$f_{eq,p} f_{eq,p'} \tilde{f}_{eq,k} \tilde{f}_{eq,k'} = f_{eq,k} f_{eq,k'} \tilde{f}_{eq,p} \tilde{f}_{eq,p'}$$

we can write the linearized collision integral in the form

$$C[f] = \frac{1}{\nu}\int dK' dP dP' \, W_{kk'\to pp'} f_{eq,k} f_{eq,k'} \tilde{f}_{eq,p} \tilde{f}_{eq,p'} \left(\phi_p + \phi_{p'} - \phi_k - \phi_{k'}\right) + O(\phi^2) \qquad (3.19)$$

- The collision term in the rank-2 e.o.m. is now

$$C_{r-1}^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} \int dK E_k^{r-1} k^\alpha k^\beta \frac{1}{2} \int dK' dP dP' W_{kk'\to pp'} f_{eq,k} f_{eq,k'} \tilde{f}_{eq,p} \tilde{f}_{eq,p'} \left(\phi_p + \phi_{p'} - \phi_k - \phi_{k'} \right) \quad (3.20)$$

- It turns out (details: arXiv:1202.4551) that we can write

$$C_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\sum_{m=0}^{\infty} \mathcal{A}_{rn}^{(\ell)} \rho_n^{\mu_1\cdots\mu_\ell} \quad (3.21)$$

where

$$\mathcal{A}_{rn}^{(\ell)} = \frac{1}{\nu(2\ell+1)} \int dK dK' dP dP' W_{kk'\to pp'} f_{eq,k} f_{eq,k'} \tilde{f}_{eq,p} \tilde{f}_{eq,p'} E_k^{r-1} k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle}$$
$$\times \left(\mathcal{H}_{kn}^{(\ell)} k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle} + \mathcal{H}_{k'n}^{(\ell)} k'_{\langle\mu_1}\cdots k'_{\mu_\ell\rangle} - \mathcal{H}_{pn}^{(\ell)} p_{\langle\mu_1}\cdots p_{\mu_\ell\rangle} - \mathcal{H}_{p'n}^{(\ell)} p'_{\langle\mu_1}\cdots p'_{\mu_\ell\rangle} \right) \quad (3.22)$$

- Essential for our purpose is that the coefficients $\mathcal{A}_{rn}^{(\ell)}$ contains all the details of the particle interactions

- In the rank-2 e.o.m. the collision term takes then a form

$$C_{r-1}^{\langle\mu\nu\rangle} = -\sum_{m=0}^{\infty} \mathcal{A}_{rn}^{(2)} \rho_n^{\mu\nu} \quad (3.23)$$

- Combining all the terms we finally get

$$\dot{\rho}_r^{\langle\mu\nu\rangle} = 2\left[I_{r+2,1} + (r-1) I_{r+2,2}\right]\sigma^{\mu\nu} - \sum \mathcal{A}_{rn}^{(2)} \rho_n^{\mu\nu} + \underbrace{\Delta^{\mu\nu}_{\alpha\beta}\int dK \frac{d}{d\tau}\left(E_k^r k^{\langle\alpha} k^{\beta\rangle}\right)\delta f - \Delta^{\mu\nu}_{\alpha\beta}\int dK E_k^r k^{\langle\alpha} k^{\beta\rangle} \frac{1}{E_k} k_\lambda \nabla^\lambda \delta f}_{\text{these lead to non-linear terms such as } \rho_r^{\mu\nu}\theta}$$

- This already resembles the Israel-Stewart equations for $\pi^{\mu\nu}$:

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{1}{\tau}\pi^{\mu\nu} = \frac{2\eta}{\tau}\sigma^{\mu\nu} + \text{higher order terms},$$

but is still a coupled equation for moments $\rho_n^{\mu_1 \dots \mu_\ell}$

- In it's full glory the rank-2 equation is

$$\begin{aligned}\dot{\rho}_r^{\langle\mu\nu\rangle} - C_{r-1}^{\langle\mu\nu\rangle} = 2\alpha_r^{(2)}\sigma^{\mu\nu} &+ \frac{2}{15}\left[(r-1)m^4\rho_{r-2} - (2r+3)m^2\rho_r + (r+4)\rho_{r+2}\right]\sigma^{\mu\nu} + \frac{2}{5}\left[r m^2 \rho_{r-1}^{\langle\mu} - (r+5)\rho_{r+1}^{\langle\mu}\right]\dot{u}^{\nu\rangle} \\ &+ r\rho_{r-1}^{\mu\nu\lambda}\dot{u}_\lambda - \frac{2}{5}\nabla^{\langle\mu}\left(m^2\rho_{r-1}^{\nu\rangle} - \rho_{r+1}^{\nu\rangle}\right) + \frac{1}{3}\left[(r-1)m^2\rho_{r-2}^{\mu\nu} - (r+4)\rho_r^{\mu\nu}\right]\theta \\ &+ \frac{2}{7}\left[(2r-2)m^2\rho_{r-2}^{\lambda\langle\mu} - (2r+5)\rho_r^{\lambda\langle\mu}\right]\sigma_\lambda^{\nu\rangle} + 2\rho_r^{\lambda\langle\mu}\omega^{\nu\rangle}{}_\lambda - \Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda \rho_{r-1}^{\alpha\beta\lambda} \\ &+ (r-1)\rho_{r-2}^{\mu\nu\lambda\kappa}\sigma_{\lambda\kappa}\end{aligned} \tag{3.24}$$

where $\alpha_r^{(2)} = I_{r+2,1} - (r-1) I_{r+2,2}$

- Similar equations can be written for all $\rho_n^{\mu_1 \cdots \mu_\ell}$

- If we recall the expansion of f_k :

$$f_k = f_{eq}\left[1 + \tilde{f}_{eq} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \mathcal{H}_n^{(\ell)} \rho_n^{\mu_1 \cdots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle}\right] \tag{3.25}$$

- Equations of motion for $\rho_n^{\mu_1 \cdots \mu_\ell}$ together with the expansion gives full dynamics of f_k

- We are still no closer to fluid dynamics, but we have re-written the Boltzmann equation in easier form (although it may not appear so)

- The main complication is that the moment e.o.m. couple all the moments: Eventually the goal is to reduce this set of equations only to the dissipative quantities, e.g. $\pi^{\mu\nu} = \rho_0^{\mu\nu}$

- For this purpose we need to be able to relate general moments $\rho_n^{\mu_1 \cdots \mu_\ell}$ to the dissipative quantities $\pi^{\mu\nu}, V^\mu, \Pi$

14-moment approximation

- If we now recall the 14-moment approximation (truncation of expansion)
  - Let's simplify a bit and assume that scalar (bulk viscosity) and vector moment (diffusion) can be neglected

$$\Rightarrow \quad f_k = f_{eq} + f_{eq}\tilde{f}_{eq} \frac{1}{2J_{42}} \pi^{\mu\nu} k_{\langle\mu} k_{\nu\rangle} \tag{3.26}$$

$$\Rightarrow \quad \rho_r = \int dK\, E_k^r\, \delta f = \underbrace{\int dK\, E_k^r\, f_{eq}\tilde{f}_{eq}\, k_{\langle\mu} k_{\nu\rangle}}_{=0} \frac{\pi^{\mu\nu}}{2J_{42}} = 0$$

$$\rho_r^{\alpha} = \int \underbrace{dK\, E_k^r\, k^{\langle\alpha\rangle} k_{\langle\mu} k_{\nu\rangle}}_{\to 0 \text{ orthogonality}} f_{eq}\tilde{f}_{eq} \frac{\pi^{\mu\nu}}{2J_{42}} = 0$$

$$\rho_r^{\alpha\beta} = \int dK\, E_k^r\, k^{\langle\alpha} k^{\beta\rangle} k_{\langle\mu} k_{\nu\rangle} f_{eq}\tilde{f}_{eq} \frac{\pi^{\mu\nu}}{2J_{42}} = \frac{J_{4+r,2}}{J_{42}} \pi^{\mu\nu}$$

every rank-2 moment is proportional to $\pi^{\mu\nu}$

$$\rho_r^{\alpha\beta\gamma} = \rho_r^{\alpha\beta\gamma\delta} = \cdots = 0$$

- If we now recall the eom, and take $r=0$ (and assume further massless gas, $m=0$)
  This is actually by Denicol, Koide & Rischke, $r=2$ is original Israel & Stewart

$$\dot{\rho}_r^{\langle\mu\nu\rangle} - C_{r-1}^{\langle\mu\nu\rangle} = 2\alpha_r^{(2)}\sigma^{\mu\nu} + \frac{2}{15}\left[(r-1)m^4\rho_{r-2} - (2r+3)m^2\rho_r + (r+4)\rho_{r+2}\right]\sigma^{\mu\nu} + \frac{2}{5}\left[r m^2\rho_{r-1}^{\langle\mu} - (r+5)\rho_{r+1}^{\langle\mu}\right]\dot{u}^{\nu\rangle}$$
$$+ r\rho_{r-1}^{\mu\nu\lambda}\dot{u}_\lambda - \frac{2}{5}\nabla^{\langle\mu}\left(m^2\rho_{r-1}^{\nu\rangle} - \rho_{r+1}^{\nu\rangle}\right) + \frac{1}{3}\left[(r-1)m^2\rho_{r-2}^{\mu\nu} - (r+4)\rho_r^{\mu\nu}\right]\theta$$
$$+ \frac{2}{7}\left[(2r-2)m^2\rho_{r-2}^{\lambda\langle\mu} - (2r+5)\rho_r^{\lambda\langle\mu}\right]\sigma_\lambda^{\nu\rangle} + 2\rho_r^{\lambda\langle\mu}\omega^{\nu\rangle}{}_\lambda - \Delta^{\mu\nu}_{\alpha\beta}\nabla_\lambda\rho_{r-1}^{\alpha\beta\lambda}$$
$$+ (r-1)\rho_{r-2}^{\mu\nu\lambda\kappa}\sigma_{\lambda\kappa}$$

- We are left with

$$\dot{\rho}_0^{\langle\mu\nu\rangle} - C_{-1}^{\langle\mu\nu\rangle} = 2\alpha_0^{(2)}\sigma^{\mu\nu} - \frac{4}{3}\rho_0^{\mu\nu}\theta - \frac{10}{7}\rho_0^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + 2\rho_0^{\lambda\langle\mu}\omega^{\nu\rangle}{}_\lambda \qquad (3.27)$$

- further $\rho_0^{\mu\nu} = \pi^{\mu\nu}$, $C_{-1}^{\langle\mu\nu\rangle} = -\mathcal{A}_{00}^{(2)}\pi^{\mu\nu}$ ← only one term in f_k expansion

$$\Rightarrow \dot{\pi}^{\langle\mu\nu\rangle} + \mathcal{A}_{00}^{(2)}\pi^{\mu\nu} = 2\alpha_0^{(2)}\sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta - \frac{10}{7}\pi^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + 2\pi^{\lambda\langle\mu}\omega^{\nu\rangle}{}_\lambda \qquad (3.28)$$

- These are the Israel-Stewart equations for $\pi^{\mu\nu}$ with $\tau = \left(\mathcal{A}_{00}^{(2)}\right)^{-1}$, $\eta = \alpha_0^{(2)}\tau$
  massless

- If take full 14-moment approximation

$$\rho_n = \gamma_n^{(0)} \Pi$$
$$\rho_n^{\mu} = \gamma_n^{(1)} V^{\mu} \qquad (3.29)$$
$$\rho_n^{\mu\nu} = \gamma_n^{(2)} \pi^{\mu\nu}$$

where $\gamma_n^{(\ell)}$'s are thermodynamical coefficients (depend on T and μ)

- Substituting these into the moment equations ($\dot{\rho}^{\langle\mu_1\cdots\mu_\ell\rangle}$) gives the full coupled Israel-Stewart equations of motion

$$\dot{\Pi} = -\frac{1}{\tau_\Pi}\left(\Pi + \zeta\theta\right) + \text{non-linear \& couplings}$$
$$\dot{V}^{\langle\mu\rangle} = -\frac{1}{\tau_V}\left(V^{\mu} - \kappa\nabla^{\mu}\left(\frac{\mu}{T}\right)\right) + \text{''} \qquad (3.30)$$
$$\dot{\pi}^{\langle\mu\nu\rangle} = -\frac{1}{\tau_\pi}\left(\pi^{\mu\nu} - 2\eta\sigma^{\mu\nu}\right) + \text{''}$$

- Is this good enough? What we want from fluid dynamical limit

  - Evolution described by the conserved currents $T^{\mu\nu}$ and N^{μ} alone

  - Fluid dynamics applicable when separation between microscopic and macroscopic scales

$$Kn = \frac{l_{micr}}{L_{macr}} \lesssim 1 \qquad \text{(Knudsen number)} \tag{3.31}$$

  - microscopic scales $\tau_\Pi, \tau_V, \tau_\pi$ macroscopic scales $\theta, \nabla^\mu\left(\frac{\mu}{T}\right), \frac{|\nabla^\mu e|}{e}$

    $\Rightarrow$ e.g. $\tau_\pi \theta \lesssim 1$

  - Fluid dynamics applicable when when close to equilibrium. Quantify by inverse Reynolds numbers

$$R_\Pi^{-1} \equiv \frac{|\Pi|}{P_{eq}} \ll 1 \qquad R_V^{-1} = \frac{|V^\mu|}{n} \ll 1 \qquad R_\pi^{-1} = \frac{|\pi^{\mu\nu}|}{P_{eq}} \ll 1 \tag{3.32}$$

  - In Israel-Stewart type of fluid dynamics these are two independent type of quantities (related by e.o.m.)

  - Want well-defined expansion in Kn and Re^{-1}

- 14-moment approximation is a direct truncation of f_k''s expansion

e.g. $\frac{\delta f}{f_{eq}} = \frac{1}{2J_{42}} \pi^{\mu\nu} k_\mu k_\nu$ + other dissipative quantities V^μ & Π + neglect $(\rho_1^{\mu\nu}, \rho_2^{\mu\nu}, \dots)$

- 14-moment approximation reduces the independent degrees of freedom to $T^{\mu\nu}$ and N^μ, but it is not truncation in Kn nor in R^{-1}

  - in 14-mom : $\rho_1^{\mu\nu} = \underbrace{\frac{J_{52}}{J_{42}}}_{\text{thermodynamic function}} \pi^{\mu\nu} \Rightarrow$ same order in R^{-1}

- Let's look at how this can be done better.

- Back to the moment equation (rank-2 as an example)

$$\underbrace{\dot{\rho}_r^{\langle\mu\nu\rangle} + \sum_{n=0}^{N_2} \mathcal{A}_{rn}^{(2)} \rho_n^{\mu\nu} = 2\alpha_r^{(2)} \sigma^{\mu\nu}}_{\text{most important part when sufficiently close to equilibrium \& gradients small}} + \text{non-linear terms}$$

$\hookrightarrow \underbrace{\text{gradient} \times \rho}_{\text{these will lead to higher-order terms } O(2) + \text{higher}}$

$O(Kn)$ theory (Navier-Stokes):

- If we can neglect time-derivative & non-linear terms $\rightarrow$

$$\sum_{n=0}^{N_L} \mathcal{A}^{(2)}_{rn}\, \rho^{\mu\nu}_n = 2\alpha^{(2)}_r \sigma^{\mu\nu} \tag{3.33}$$

- Define inverse of $\mathcal{A}^{(2)}_{rn}$: $\sum_r \tau^{(2)}_{mr} \mathcal{A}^{(2)}_{rn} = \delta_{mn}$ (3.34)

$$\Rightarrow \sum_{rn} \tau^{(2)}_{mr} \mathcal{A}^{(2)}_{rn}\, \rho^{\mu\nu}_n = \sum_r 2\alpha_r \tau^{(2)}_{mr}\, \sigma^{\mu\nu} \tag{3.35}$$

$$\Rightarrow \quad \delta_{mn}\rho^{\mu\nu}_n = \sum_r 2\alpha^{(2)}_r \tau^{(2)}_{mr}\, \sigma^{\mu\nu} \tag{3.36}$$

$$\xRightarrow{m=0} \quad \rho^{\mu\nu}_0 = \pi^{\mu\nu} = 2 \underbrace{\sum_r \tau^{(2)}_{0r} \alpha^{(2)}_r}_{\text{This can be identified as shear viscosity } \eta} \sigma^{\mu\nu} \tag{3.37}$$

- Shear viscosity given by the inverse of the collision matrix $\mathcal{A}^{(2)}_{rn}$
  - compare to 14-mom. approximation $\eta = \left(\mathcal{A}^{(2)}_{00}\right)^{-1} \alpha^{(2)}_0$
    $\hookrightarrow$ just one term from the full matrix

- In order to get the full 1st-order theory we need to sum all moments $\rho_n^{\mu\nu}$

  ▷ 14-mom. approximation neglects infinitely many $O(Kn)$ terms

- Same applies to the transient fluid dynamics (where $2\eta\sigma^{\mu\nu}$ is one of the terms)

Transient fluid dynamics

- Restore time derivative to moment eqs.

$$\dot{\rho}_r^{\langle\mu\nu\rangle} + \sum_{n=0}^{N_2} \mathcal{A}_{rn}^{(2)} \rho_n^{\mu\nu} = 2\alpha_r^{(2)} \sigma^{\mu\nu} + \underbrace{\text{non-linear terms}}_{\text{ignore these for now}} \tag{3.38}$$

- Linearized moment equation is now

$$\dot{\rho}_r^{\langle\mu\nu\rangle} + \sum_n \mathcal{A}_{rn}^{(2)} \rho_n^{\mu\nu} = 2\alpha_r^{(2)} \sigma^{\mu\nu} \tag{3.39}$$

- Diagonalize this system

• Define matrix $\Omega^{(2)}$ that diagonalizes the collision matrix $\mathcal{A}^{(2)}$

$$(\Omega^{(2)})^{-1}\mathcal{A}^{(2)}\Omega^{(2)} = \mathrm{diag}\left(\chi_0^{(2)}, \chi_1^{(2)}, \dots, \chi_{N_2}^{(2)}\right) \tag{3.40}$$

• χ_i are the eigenvalues of $\mathcal{A}^{(2)}$

• Define further rank-2 tensors

$$X_i^{\mu\nu} = \sum_j (\Omega^{(2)})^{-1}_{ij}\, \rho_j^{\mu\nu} \tag{3.41}$$

∘ these are the eigenmodes of (linearized) Boltzmann equation

⇒ Linearized moment equation:

$$\dot{\rho}_r^{\langle\mu\nu\rangle} + \sum_n \mathcal{A}^{(2)}_{rn} \underset{\Omega\Omega^{-1}}{\wedge} \rho_n^{\mu\nu} = 2\alpha_r^{(2)}\sigma^{\mu\nu} \qquad \Big|\ \Omega^{-1}\times \tag{3.42}$$

time derivative of Ω

$$\Rightarrow \dot{X}_i^{\langle\mu\nu\rangle} + \chi_i^{(2)} X_i^{\mu\nu} = \beta_i^{(2)}\sigma^{\mu\nu} + \text{higher-order terms} \tag{3.43}$$

$$\Big\|\ \beta_i^{(2)} = 2\sum_j (\Omega^{(2)})^{-1}_{ij}\,\alpha_j^{(2)}$$

$\Rightarrow$ Equations of motion for $X_i^{\mu\nu}$ decouple (in linear regime)

$1/x_i^{(2)}$ are now relaxation times for tensors $X_i^{\mu\nu}$ (eigenmodes of $\mathcal{A}^{(2)}$)

- Note that if the gradients $\sigma^{\mu\nu} = 0$

$$\Rightarrow \dot{X}_i^{\langle\mu\nu\rangle} + x_i^{(2)} X_i = 0 \quad \Longrightarrow \quad X_i^{\mu\nu} \xrightarrow{1/x_i^{(2)}} 0 \qquad (3.44)$$

$\Rightarrow$ $1/x_i^{(2)}$ are thermalization times of the system

- If we wait long enough $\Rightarrow$ slowest thermalization / relaxation time dominates!

- Order $x_i^{(2)}$'s such that $1/x_0^{(2)}$ is the slowest time scale

- If we now assume that only the slowest mode is fully dynamical

$$\dot{X}_0^{\langle\mu\nu\rangle} + x_0^{(2)} X_0^{\mu\nu} = \beta_0^{(2)} \sigma^{\mu\nu} + \text{higher-order} \qquad (3.45)$$

and rest of the modes can be approximated as

$$X_r^{\mu\nu} = \frac{\beta_r^{(2)}}{x_r^{(2)}} \sigma^{\mu\nu} + \text{higher-order} \qquad (\text{for } r \neq 0) \qquad (3.46)$$

$\Rightarrow$ We have reduced the independent degrees of freedom to $X_0^{\mu\nu}$

- Good, but we still need to express everything in terms of $\pi^{\mu\nu}$, V^μ and Π

- We can first invert $\quad X_i^{\mu\nu} = \sum_j \left(\Omega^{(2)}\right)^{-1}_{ij} \rho_j^{\mu\nu}$

$$\Rightarrow \rho_i^{\mu\nu} = \sum_j \Omega^{(2)}_{ij} X_j^{\mu\nu} = \Omega^{(2)}_{i0} X_0^{\mu\nu} + \sum_{j=1} \Omega^{(2)}_{ij} \frac{\beta_j^{(2)}}{\chi_j^{(2)}} \sigma^{\mu\nu} = \Omega^{(2)}_{i0} X_0^{\mu\nu} + O(Kn) \qquad (3.47)$$

- take $i=0$, so that $\rho_0^{\mu\nu} = \pi^{\mu\nu}$, and set $\Omega^{(2)}_{00} = 1$

$$\Rightarrow X_0^{\mu\nu} = \pi^{\mu\nu} - \sum_{j=1} \Omega^{(2)}_{ij} \frac{\beta_j^{(2)}}{\chi_j^{(2)}} \sigma^{\mu\nu}$$

substitute

$$\Rightarrow \rho_i^{\mu\nu} \simeq \Omega^{(2)}_{i0} \pi^{\mu\nu} - \sum_{j=1} \Omega^{(2)}_{i0} \Omega^{(2)}_{ij} \frac{\beta_j^{(2)}}{\chi_j^{(2)}} \sigma^{\mu\nu} + \sum_{j=1} \Omega^{(2)}_{ij} \frac{\beta_j^{(2)}}{\chi_j^{(2)}} \sigma^{\mu\nu} + \text{H.O.} \qquad (3.48)$$

|| use: $\beta_i^{(2)} = 2 \sum_j \left(\Omega^{(2)}\right)^{-1}_{ij} \alpha_j^{(2)}$ & $\tau^{(2)}_{in} = \sum_{m=0}^{N_\ell} \Omega^{(\ell)}_{im} \frac{1}{\chi_m^{(\ell)}} \left(\Omega^{-1}\right)^{(\ell)}_{mn}$

↑ inverse of diagonal matrix $\begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & \ddots \end{pmatrix} = \begin{pmatrix} 1/a & & \\ & 1/b & \\ & & 1/c \end{pmatrix}$

- Finally we get the relation

$$\rho_i^{\mu\nu} = \Omega_{i0}^{(2)} \pi^{\mu\nu} + \left(2\eta_i - \Omega_{i0}^{(2)} \eta_0\right) \sigma^{\mu\nu} = \Omega_{i0}^{(2)} \pi^{\mu\nu} + O(Kn) \qquad (3.49)$$

where $\eta_i = \sum_{r=0} \tau_{ir}^{(2)} \alpha_r^{(2)}$, so that $\eta_0 = \eta$ = shear viscosity

- These are now relations that can be substituted into the original moment equations (similar relations can be derived for ρ_i and ρ_i^μ as well)

$$\begin{aligned}\dot{\rho}_r^{\langle\mu\nu\rangle} - C_{r-1}^{\langle\mu\nu\rangle} = 2\alpha_r^{(2)} \sigma^{\mu\nu} &+ \frac{2}{15}\left[(r-1) m^4 \rho_{r-2} - (2r+3) m^2 \rho_r + (r+4) \rho_{r+2}\right] \sigma^{\mu\nu} + \frac{2}{5}\left[r m^2 \rho_{r-1}^{\langle\mu} - (r+5) \rho_{r+1}^{\langle\mu}\right] \dot{u}^{\nu\rangle} \\ &+ r \rho_{r-1}^{\mu\nu\lambda} \dot{u}_\lambda - \frac{2}{5} \nabla^{\langle\mu} \left(m^2 \rho_{r-1}^{\nu\rangle} - \rho_{r+1}^{\nu\rangle}\right) + \frac{1}{3}\left[(r-1) m^2 \rho_{r-2}^{\mu\nu} - (r+4) \rho_r^{\mu\nu}\right] \theta \\ &+ \frac{2}{7}\left[(2r-2) m^2 \rho_{r-2}^{\lambda\langle\mu} - (2r+5) \rho_r^{\lambda\langle\mu}\right] \sigma_\lambda^{\nu\rangle} + 2\rho_r^{\lambda\langle\mu} \omega^{\nu\rangle}{}_\lambda - \Delta_{\alpha\beta}^{\mu\nu} \nabla_\lambda \rho_{r-1}^{\alpha\beta\lambda} \\ &+ (r-1) \rho_{r-2}^{\mu\nu\lambda\kappa} \sigma_{\lambda\kappa} \qquad (3.50)\end{aligned}$$

- Note that there are some moments $\rho_i^{\mu\nu}$ in the equations for which $i < 0$, and above relation is only for $i \geq 0$

$\Rightarrow$ We can use $\rho_{-i}^{\mu\nu} = \int dK\, E^{-i} k^{\langle\mu} k^{\nu\rangle} \delta f$ (3.51)

substitute expansion with $\rho_i^{\mu\nu}$'s, $i \geq 0$ into here

- The form of the final equation for $\pi^{\mu\nu}$ is now

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + K^{\mu\nu} + R^{\mu\nu}, \quad (3.52)$$

where $\tau_\pi = 1/\chi_{(0)}^{(2)}$, $\eta = \sum_r \tau_{0r}^{(2)} \alpha_r^{(2)}$

- $\mathcal{J}^{\mu\nu}$ contains terms of order $Kn \times R^{-1}$

$$\mathcal{J}^{\mu\nu} = 2\pi_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}_\lambda + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \tau_{\pi V} V^{\langle\mu}\nabla^{\nu\rangle}p_{eq} + \ell_{\pi V}\nabla^{\langle\mu}V^{\nu\rangle} + \lambda_{\pi n} V^{\langle\mu}\nabla^{\nu\rangle}\frac{\mu}{T}$$

- $K^{\mu\nu}$ terms Kn^2 (absent completely in Israel-Stewart theory)

$$K^{\mu\nu} = \eta_1 \omega_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_2\theta\sigma^{\mu\nu} + \eta_3\sigma^{\lambda\langle\mu}\sigma^{\nu\rangle}_\lambda + \eta_4\sigma_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_5\nabla^{\langle\mu}\left(\frac{\mu}{T}\right)\nabla^{\nu\rangle}\left(\frac{\mu}{T}\right)$$
$$+ \eta_6\nabla^{\langle\mu}p_{eq}\nabla^{\nu\rangle}p_{eq} + \eta_7\nabla^{\langle\mu}\left(\frac{\mu}{T}\right)\nabla^{\nu\rangle}p_{eq} + \eta_8\nabla^{\langle\mu}\nabla^{\nu\rangle}\left(\frac{\mu}{T}\right) + \eta_9\nabla^{\langle\mu}\nabla^{\nu\rangle}p_{eq}$$

- $R^{\mu\nu}$ contains $(R^{-1})^2$ that come from the non-linear part of the collision integral

$$R^{\mu\nu} = \varphi_6 \Pi \sigma^{\mu\nu} + \varphi_7 \pi^{\lambda\langle\mu}\pi^{\nu\rangle}_{\lambda} + \varphi_8 V^{\langle\mu}V^{\nu\rangle}$$

- Every type of term that can appear will appear

- These are the DNMR equations. (arXiv: 1202.4551)

(all the moments summed into coefficients, slowest microscopic time identified as relaxation time, Kn and R^{-1} identified as separate quantities)

- 14-mom. approximation

$$\frac{\delta f}{f_k} = \varepsilon + \varepsilon^{\mu} k_\mu + \varepsilon^{\mu\nu} k_\mu k_\nu$$

e.g. $\rho_n^{\mu\nu} = \Omega^{(2)}_{i0} \pi^{\mu\nu} + O(Kn)$

full ex. $\Longrightarrow$ $f_k = f_{eq}\left[1 + \tilde{f}_{eq} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \mathcal{H}^{(\ell)}_{n} \rho_n^{\mu_1 \cdots \mu_\ell} k_{\langle\mu_1} \cdots k_{\mu_\ell\rangle}\right]$

- Note: $\rho_i^{\mu\nu\alpha} / \rho_i^{\mu\nu\alpha\beta} \sim O(2)$, first order term doesn't exist

- Problem are Kn^2 terms & contains e.g. $(\partial_t)^2$ terms
  $\Rightarrow$ Potentially ruin causality of the theory

Other approaches

- In DNMR approach

$$\rho_r^{\mu\nu} = \Omega_{r0}^{(2)} \pi^{\mu\nu} + O(Kn)$$

This is the reason for $O(Kn^2)$ terms

- Instead of trying to identify relaxation time as the real slowest timescale of BE, assume that we are near asymptotic regime where

$$\rho_i^{\mu\nu} = 2\eta_i \sigma^{\mu\nu} + O(2)$$

"shear viscosity of moment $\rho_i^{\mu\nu}$, $\eta_0 = \eta$

$$\Rightarrow \left. \begin{aligned} \pi^{\mu\nu} &= 2\eta\sigma^{\mu\nu} + O(2) \\ \rho_i &= 2\eta_i \sigma^{\mu\nu} + O(2) \end{aligned} \right\} \Rightarrow \rho_i^{\mu\nu} = \frac{\eta_i}{\eta} \pi^{\mu\nu} + O(2)$$

- Order-of-magnitude approximation (Fotakis talk)

- We can then substitute $\rho_i^{\mu\nu} = \frac{\eta_i}{\eta} \pi^{\mu\nu} + \underbrace{O(2)}_{\text{Can be neglected as gives at least } O(3) \text{ terms in eom's}}$ into moment equations

- Nice feature is that $O(Kn^2)$ terms are now absent.

- Relaxation time no longer slowest timescale of DE, but some effective timescale

- Re^{-1} and Kn are not anymore completely independent, but are of the same order

$$\underbrace{\frac{\pi^{\mu\nu}}{P_{eq}}}_{R^{-1}} \sim \underbrace{\frac{2\eta}{P_{eq}} \sigma^{\mu\nu}}_{Kn} + O(2)$$

  - In principle there is no such restriction in DNMR, as long as Kn and R^{-1} sufficiently small

- It is possible to have both identification of slowest timescale and absense of explicit Kn^2 terms by doubling d.o.f

$$\rho_r^{\mu\nu} = \#_{0r}\pi^{\mu\nu} + \#_{1r}\rho_1^{\mu\nu} + O(Kn^2)$$

↑ new degree of freedom in fluid dynamics

$$\Longrightarrow \begin{cases} \dot{\pi}^{\langle\mu\nu\rangle} + \tau_{00}\pi^{\mu\nu} + \tau_{01}\rho_1^{\mu\nu} = 2\alpha^{(0)}\sigma^{\mu\nu} + \text{higher-order} \\ \dot{\rho}_1^{\langle\mu\nu\rangle} + \tau_{10}\pi^{\mu\nu} + \tau_{11}\rho_1^{\mu\nu} = 2\alpha^{(1)}\sigma^{\mu\nu} + \text{higher-order} \end{cases}$$

→ no $O(Kn^2)$ terms

▲ This is in good agreement with direct numerical solutions of BE (more transport coef. & more initial conditions)

Bouras et. al, arXiv: 1207.6811

Generalization to multicomponent system (Fotakis et. al. 2203.11549)

- BE $k^\mu \partial_\mu f_i = C[\{f_i\}]$

  for each species i

$\Rightarrow$ moment equations for $\dot{\rho}_{i,r}^{\langle\mu\nu\rangle} = \sum \# \rho_{i,n}^{\mu\nu} + 2\alpha_{i,r}^{(0)} \sigma^{\mu\nu}$ + non-linear

collision integral
sum over moments & particles

particle, moment

- Order-of-magnitude : $\rho_{i,r}^{\mu\nu} = \frac{\eta_{i,r}}{\eta} \pi^{\mu\nu} + O(2)$

  $\hookrightarrow$ total $\pi^{\mu\nu} = \sum_i \pi_i^{\mu\nu}$

$\eta_{i,r}$ = "shear viscosity" of particle i, moment r

- $T^{\mu\nu} = \sum_i T_i^{\mu\nu} = e u^\mu u^\nu - (p_{eq} + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$

  matching $e = e_{eq}$, but $e = \sum_i (e_{eq,i} + \delta e_i)$

  matching $\delta e = \sum_i \delta e_i = 0$, but $\delta e_i \neq 0$

  $\hookrightarrow$ determined dynamically $\propto \Pi$

- $N_B^\mu = \sum_i B_i N_i^\mu = n_B u^\mu + V_B^\mu$ & N_S^μ and N_Q^μ

  matching $n_{B,eq} = n_B = \sum_i B_i (n_i + \delta n_i)$, $\sum_i B_i \delta n_i = 0$, $\delta n_i \neq 0$

- $n_{B,eq}(T, \mu_B, \mu_S, \mu_Q) = n_B$ $n_{S,eq}(T, \mu_B, \mu_S, \mu_Q) = n_S$ $n_{Q,eq}(T, \mu_B, \mu_S, \mu_Q) = n_Q$

  & $\varepsilon_{eq}(T, \mu_B, \mu_S, \mu_Q) = \varepsilon$

$$\Downarrow$$

- Single fluid with $\underbrace{u^\mu, T}_{\text{common } u^\mu \,\&\, T}, \{\mu_i = B_i\mu_B + S_i\mu_S + Q_i\mu_Q\}$