Quantum effects in the fireball and nontrivial vacuum

Outline

- Quantum corrections to the classical free streaming
- Off shell hydrodynamic expansion
- Particles production from a non-trivial vacuum

Leonardo Tinti, 26/4/'24,

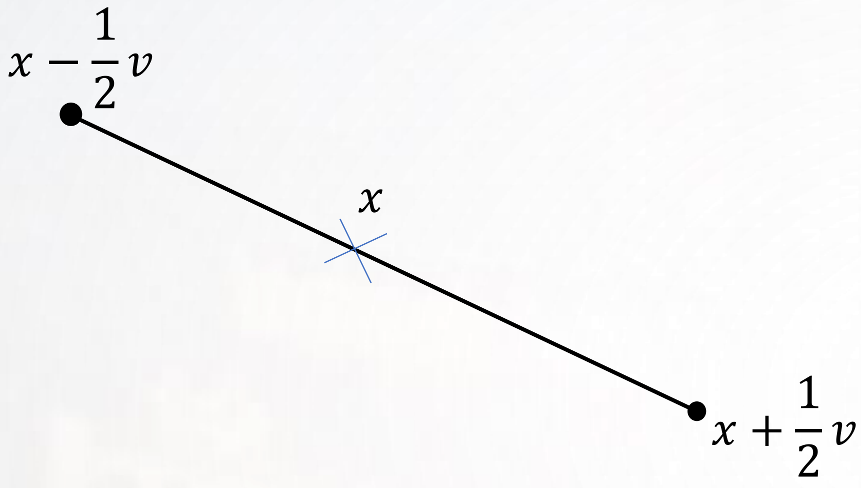
Various Faces of QCD

Uniwersytet Jana Kochanowskiego w Kielcach

LT, 10.1103/PhysRevD.108.076022, 10.1103/PhysRevD.108.036015
LT, A Vereijken, S Jafarzade, F Giacosa, arxiv:2403.15531

Narodowe Centrum Nauki

The link between quantum fields and relativistic kinetic theory



$$W(x,k) \propto \int \frac{d^4v}{(2\pi)^4} e^{-ik\cdot v} \langle \Phi^\dagger\,(x + 1/2v)\Phi(x - 1/2v)\rangle$$

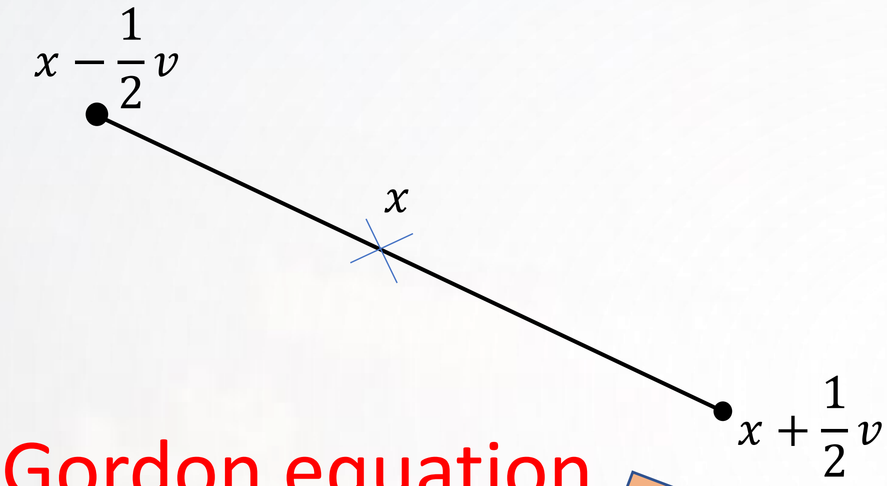
$$T^{\mu\nu}(x) = \int d^4k\;\; k^\mu k^\nu W(x,k)$$

$$W(x,k) \longrightarrow (2\pi)\delta(p^2 - m^2) f_{cl.}(x,p)$$

$$k \cdot \partial W(x,k) = \cdots \longrightarrow p \cdot \partial f(x,p) = \cdots$$

- Relativistic Kinetic Theory. Principles and Applications - De Groot, S.R. et al. Amsterdam, Netherlands: North-Holland (1980)

The link between quantum fields and relativistic kinetic theory



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Klein-Gordon equation

$$\left[\frac{1}{4}\hbar^2\Box - \left(k^2 - m^2c^2\right) + i\hbar k\cdot\partial\right] W(x,k) = \cdots$$

- T. S. Biro and A. Jakovac, *Emergence of Temperature in Examples and Related Nuisances in Field Theory*, Springer Briefs in Physics (2019)

Quantum free streaming

Exact solutions in 1+1 dimensions

$w = zk^0 - tk^z$

$$W(t,z;k^0,k_T,k^z) = \delta(k^0)\delta(k^z)\int d\xi\left[e^{-i\left(t\sqrt{4m_T^2+\xi^2}-z\,\xi\right)}\mathcal{A}(\xi;k_T) + e^{i\left(t\sqrt{4m_T^2+\xi^2}-z\,\xi\right)}\mathcal{A}^*(\xi;k_T)\right]$$

$$+\cos\left(2w\sqrt{\frac{k^2-m^2}{(k^0)^2-(k^z)^2}}\right)\mathcal{F}_{\rm even}(k_0,k_T,k^z) + \sqrt{\frac{(k^0)^2-(k^z)^2}{k^2-m^2}}\,\sin\left(2w\sqrt{\frac{k^2-m^2}{(k^0)^2-(k^z)^2}}\right)\mathcal{F}_{\rm odd}(k_0,k_T,k^z)$$

Proper classical limit

$$T^{\mu\nu}(x) = \int d^4k\; k^\mu k^\nu W(x,k) \xrightarrow{\text{small }\hbar} \int \frac{d^3p}{(2\pi\hbar)^3 E_{\mathbf{p}}/c}\, p^\mu p^\nu \left[f(x,\mathbf{p}) + \bar{f}(x,\mathbf{p})\right],$$

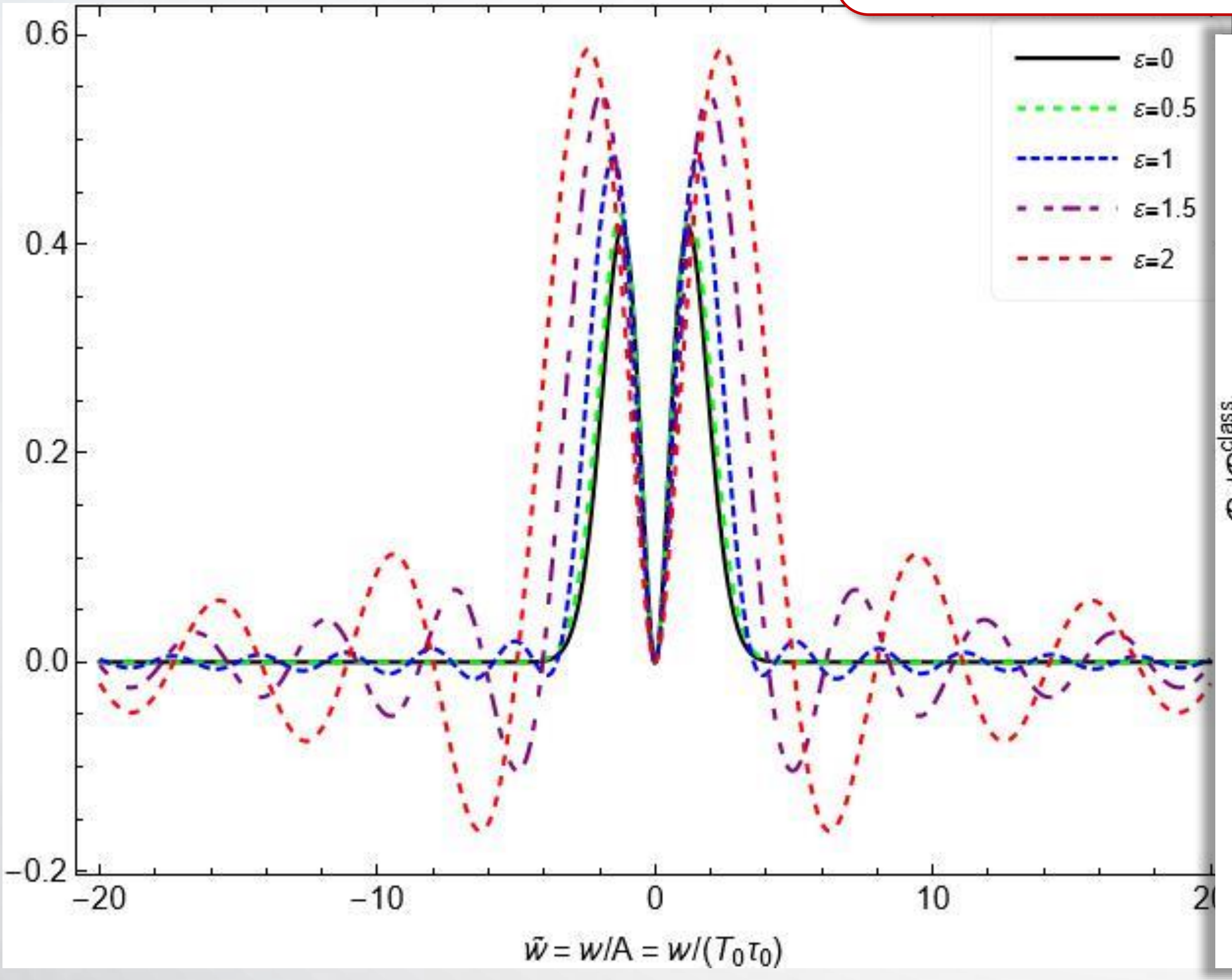
$$J^\mu = \int d^4k\; k^\mu W(x,k) \xrightarrow{\text{small }\hbar} \int \frac{d^3p}{(2\pi\hbar)^3 E_{\mathbf{p}}/c}\, p^\mu \left[f(x,\mathbf{p}) - \bar{f}(x,\mathbf{p})\right].$$

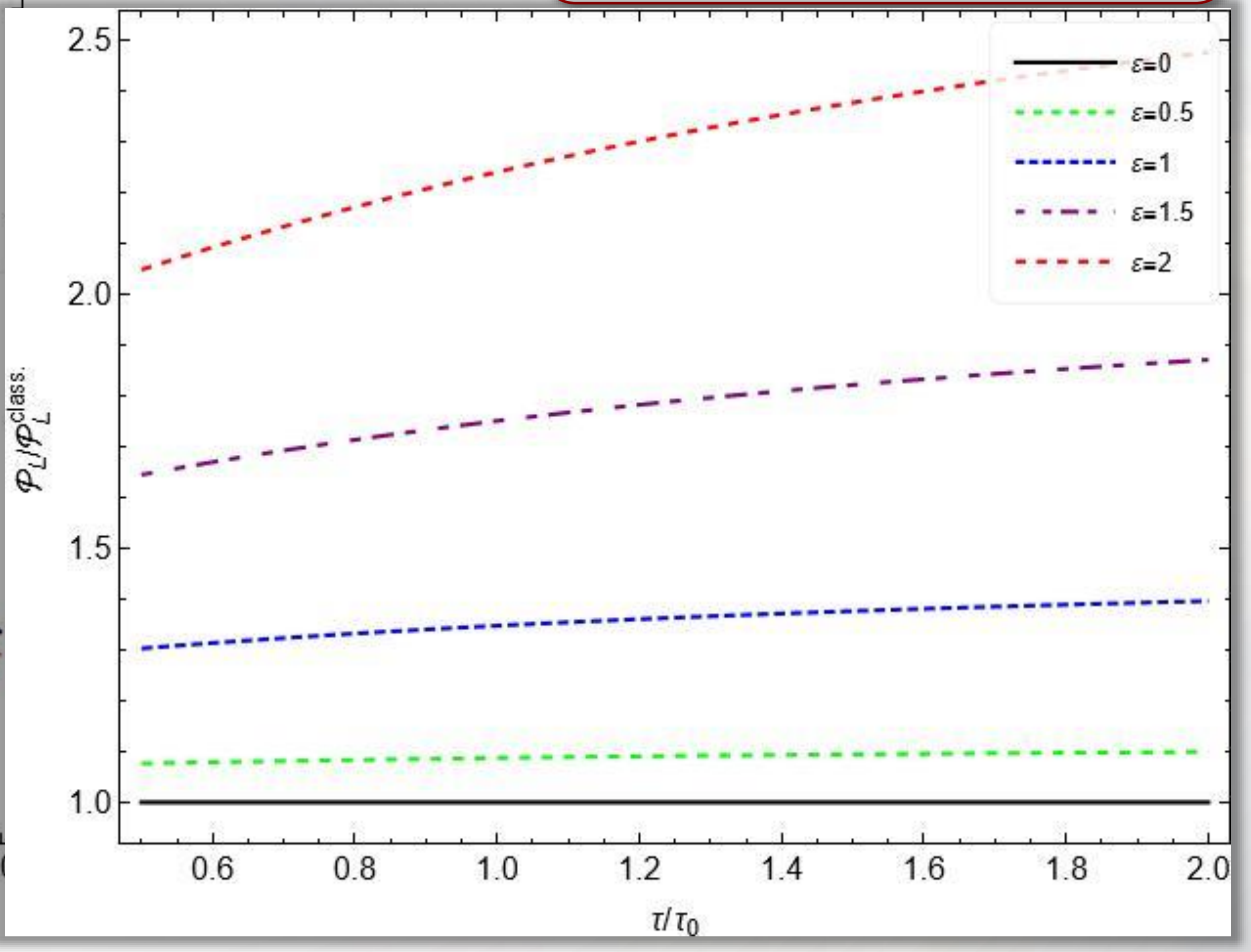
10.1103/PhysRevD.108.076022

Simple case: numerical results

$$A = T_0 \tau_0\,, \qquad \varepsilon = \frac{\hbar}{A}, \qquad \tilde{w} = \frac{w}{A}$$

$$\tilde{f}_{\text{even}} = 2\sqrt{2\pi}\,\frac{\pi^4}{30}\,\exp\left\{-\frac{k_T^2}{2T_0^2}\frac{4}{4-\chi^2} - \frac{\chi^2}{2\varepsilon^2}\right\} \quad \Rightarrow \quad f(w,k_T) = \frac{\pi^4}{30}\,\exp\left\{-\frac{k_T^2}{2T_0^2} - \frac{w^2}{2T_0^2\tau_0^2}\right\}$$



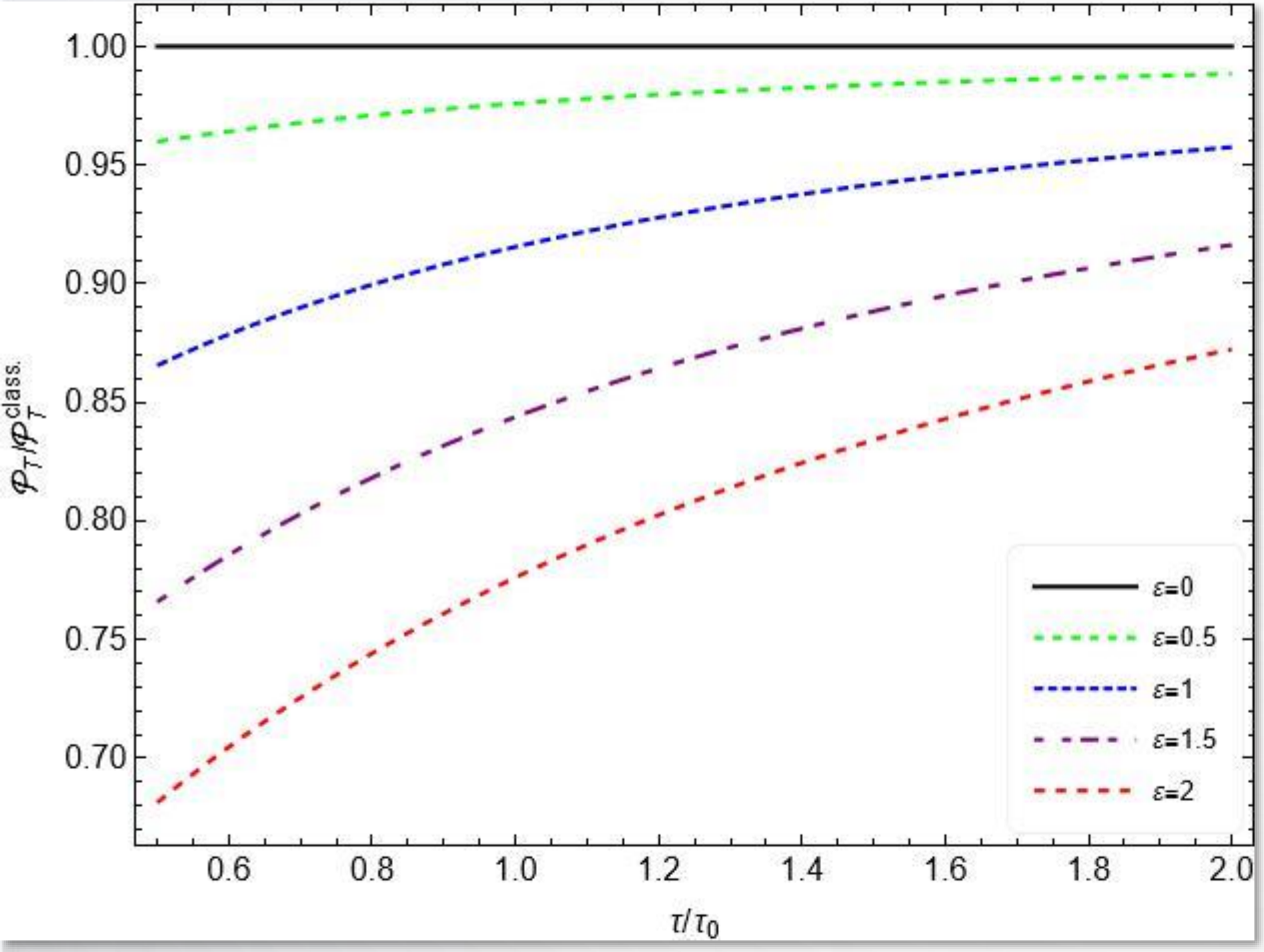


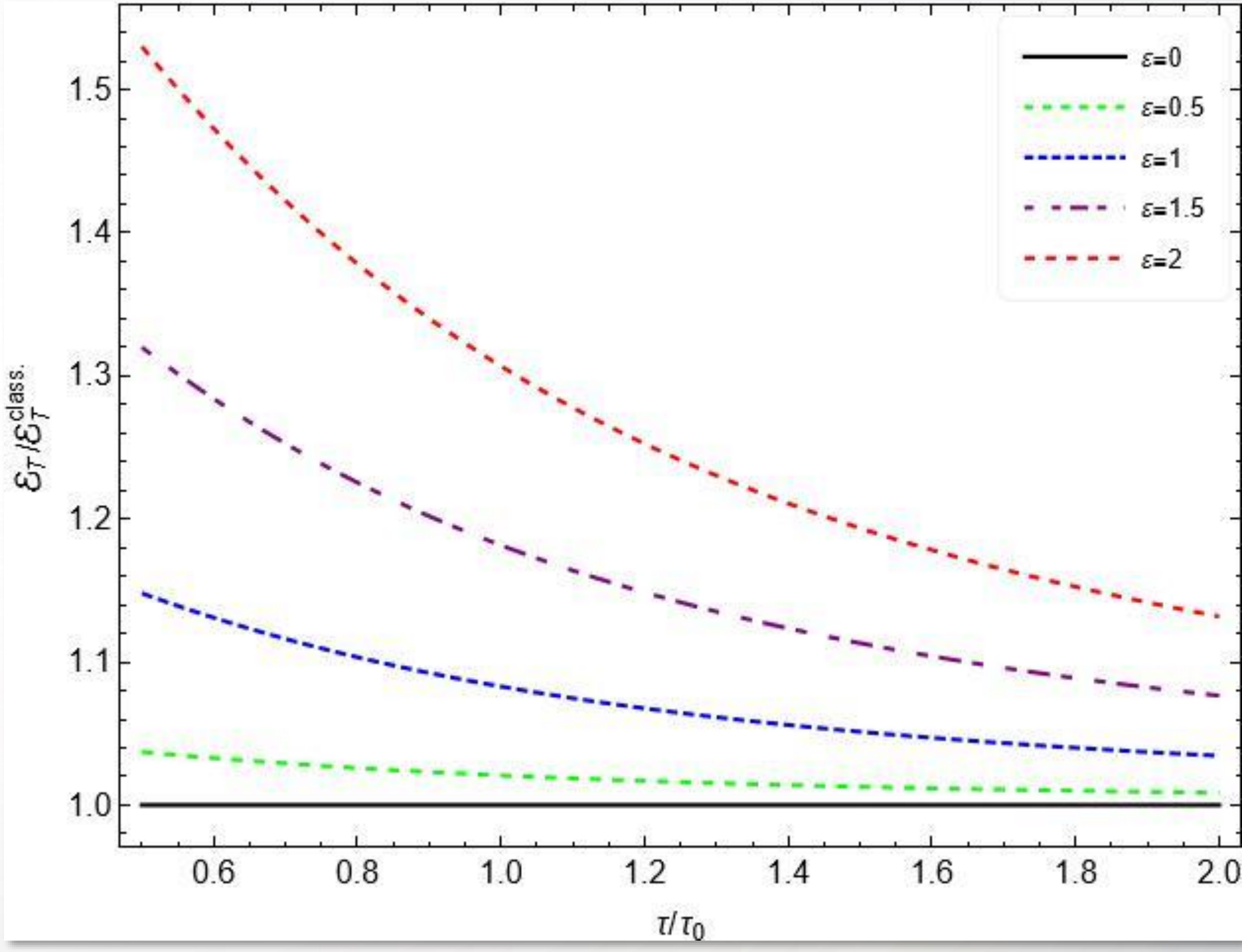
The (non-trivial part of the) integrand of $\mathcal{P}_L$

Simple case: numerical results

$$A = T_0 \tau_0\,, \quad \varepsilon = \frac{\hbar}{A}, \quad \widetilde{w} = \frac{w}{A}$$

Then the energy density and the transverse pressure





Interacting case: Hydrodynamic expansion:
(classical, method of moments)

Relativistic Boltzmann equation

$$p \cdot \partial f = -\mathcal{C}[f]$$

$$\Rightarrow \int_{\boldsymbol{p}} p^\nu p \cdot \partial f = -\int_{\boldsymbol{p}} p^\nu C = 0$$

$$\partial_\mu T^{\mu\nu}$$

$$u \cdot \partial f = \dot{f} = -\frac{p \cdot \nabla f}{(p \cdot u)} - \frac{C[f]}{(p \cdot u)}$$

extra needed equations

$$\dot{T}^{\mu\nu} = \int_{\boldsymbol{p}} p^\mu p^\nu \dot{f}$$

covariant momentum integral

$$\int_{\mathbf{p}} = \int d^4p \, 2\Theta(p_0)\delta(p^2 - m^2)$$

G Denicol, J.Phys. G41 (2014) no.12, 124004

What if we could use it? (in heavy-ion collisions)

$\mathcal{O}^{\langle\mu_1\rangle\cdots\langle\mu_l\rangle} = \Delta^{\mu_1}_{\alpha_1}\cdots\Delta^{\mu_l}_{\alpha_l}\mathcal{O}^{\alpha_1\cdots\alpha_l}$ a convenient basis

$$\mathfrak{f}_r^{\mu_1\cdots\mu_l} = \int_p (p\cdot u)^r \, p^{\langle\mu_1\rangle\cdots\langle\mu_l\rangle} f$$

and a popular decomposition of the degrees of freedom

$$\partial_\mu u_\nu = u_\mu \dot{u}_\nu + \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3}\theta\Delta_{\mu\nu}, \qquad T^{\mu\nu} = \mathcal{E}u^\mu u^\nu + \mathcal{P}^{\mu\nu} = \mathcal{E}u^\mu u^\nu - (\mathcal{P}+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

lots of self interactions in the exact evolution

$$\dot{\mathcal{P}}^{\langle\mu\rangle\langle\nu\rangle} + C_{-1}^{\langle\mu\rangle\langle\nu\rangle} = 2(\mathcal{P}+\Pi)\sigma^{\mu\nu} + \frac{5}{3}\theta(\mathcal{P}+\Pi)\Delta^{\mu\nu} - \frac{5}{3}\theta\pi^{\mu\nu} - 2\pi_\alpha^{(\mu}\sigma^{\nu)\alpha} + 2\pi_\alpha^{(\mu}\omega^{\nu)\alpha}$$
$$-\nabla_\alpha \mathfrak{f}_{-1}^{\alpha\langle\mu\rangle\langle\nu\rangle} - \left(\sigma_{\alpha\beta} + \frac{1}{3}\theta\Delta_{\alpha\beta}\right)\mathfrak{f}_{-2}^{\alpha\beta\mu\nu}$$

Problematic generalization to the Wigner distribution

$$\delta(p^2 - m^2) f \to W$$

$$p \cdot \partial f \to k \cdot \partial W$$

different physical situations
very similar to kinetic theory but…

…ill defined non-hydrodynamic tensors!

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^{\langle\alpha\rangle} k^{\langle\mu\rangle} k^{\langle\nu\rangle}}{(k \cdot u)} W \int \frac{d^4k}{(2\pi)^4} \frac{k^{\langle\alpha\rangle} k^{\langle\beta\rangle} k^{\langle\mu\rangle} k^{\langle\nu\rangle}}{(k \cdot u)^2} W$$

Solution: add a $(\boldsymbol{k} \cdot \boldsymbol{u})^n e^{-\zeta(\mathrm{k}\cdot\boldsymbol{u})^2}$ to the definition of the moments (to keep the power of $(\boldsymbol{k} \cdot \boldsymbol{u})$ positive)

$$\phi_n^{\mu_1 \cdots \mu_s}(x, \zeta) = \int \frac{d^4k}{(2\pi)^4} (k \cdot u)^n e^{-\zeta(k\cdot u)^2} k^{\langle\mu_1\rangle} \dots k^{\langle\mu_s\rangle} W(x, k)$$

Then

$$\dot{\mathcal{P}}^{\langle\mu\rangle\langle\nu\rangle} = 2(\mathcal{P} + \Pi)\sigma^{\mu\nu} + \frac{5}{3}\theta(\mathcal{P} + \Pi)\Delta^{\mu\nu} - \frac{5}{3}\theta\pi^{\mu\nu} - 2\pi_\alpha^{(\mu}\sigma^{\nu)\alpha} + 2\pi_\alpha^{(\mu}\omega^{\nu)\alpha}$$
$$+ \int_0^\infty d\zeta \left\{ \tilde{C}_1^{\langle\mu\rangle\langle\nu\rangle} - \nabla_\alpha \phi_1^{\alpha\langle\mu\rangle\langle\nu\rangle} + \dot{u}_\alpha \left[2\phi_1^{\alpha\mu\nu} - 2\zeta\phi_3^{\alpha\mu\nu}\right] + \nabla_\alpha u_\beta \left[\phi_0^{\alpha\beta\mu\nu} - 2\zeta\phi_2^{\alpha\beta\mu\nu}\right] \right\}$$

10.1103/PhysRevD.108.036015

Hydrodynamic expansion

$$\hat{\mathcal{L}}\,[f] = 2\zeta\, f(\zeta) - \int_{\zeta}^{\infty} d\zeta' f(\zeta')$$

$$\mathcal{E} = L_0(\tau, \zeta = 0)$$

$$\mathrm{P}_L = \int_{\zeta}^{\infty} d\zeta'\, L_1(\tau, \zeta')$$

$$\mathrm{P}_T = \int_{\zeta}^{\infty} d\zeta'\, T_0(\tau, \zeta')$$

Hydrodynamics

$$\dot{\mathcal{E}} = -\frac{\mathcal{E} + \mathrm{P}_L}{\tau}$$

$$\dot{\mathrm{P}}_L + \frac{1}{\tau_R}(\mathrm{P}_L - \mathrm{P}) = -\frac{3}{\tau}\mathrm{P}_L + \frac{1}{\tau}\mathcal{R}_L^{(1)}$$

$$\dot{\mathrm{P}}_T + \frac{1}{\tau_R}(\mathrm{P}_T - \mathrm{P}) = -\frac{1}{\tau}\mathrm{P}_T + \frac{1}{\tau}\mathcal{R}_T^{(1)}$$

systematically improvable set of scalar equations…

$$\mathcal{R}_T^{(n)} = \int_0^{\infty} d\zeta\, \left(\hat{\mathcal{L}}\right)^n T_n\,, \qquad \mathcal{R}_L^{(n)} = \int_0^{\infty} d\zeta\, \left(\hat{\mathcal{L}}\right)^n L_{n+1}$$

$$\dot{\mathcal{R}}_T^{(n)} + \frac{1}{\tau_R}\delta\mathcal{R}_T^{(n)} = -\frac{2n+1}{\tau}\mathcal{R}_T^{(n)} + \frac{1}{\tau}\mathcal{R}_T^{(n+1)}$$

$$\dot{\mathcal{R}}_L^{(n)} + \frac{1}{\tau_R}\delta\mathcal{R}_T^{(n)} = -\frac{2n+3}{\tau}\mathcal{R}_L^{(n)} + \frac{1}{\tau}\mathcal{R}_L^{(n+1)}$$

…to test against the exact solutions

10.1103/PhysRevD.108.036015

Exact solutions for the Wigner distribution

- Conformal equation of state (equilibrium), $W_{eq.} = \frac{2\delta(k^2)}{(2\pi)^3} e^{-\frac{1}{T(\tau)}\sqrt{k_T^2+\frac{w^2}{\tau^2}}}$
- Constant shear-viscosity over entropy ratio: $\tau_R = 5\bar{\eta}/\mathrm{T}$
- $\bar{\eta} = 3/(4\pi)$
- $\tau_0 = 1/4$ fm/c, $\mathrm{T}_0 = 0.6$ GeV, two possible initial conditions:

$$W_0^{iso} = \frac{2}{(2\pi)^3\sqrt{2\pi\sigma}} \mathrm{e}^{-\frac{v^2}{2\tau_0^2\sigma}} e^{-\frac{1}{T_0}\sqrt{\sigma=k_T^2+\frac{w^2}{\tau_0^2}}} \longrightarrow \mathcal{P}_0 = \mathcal{P}_{\mathrm{eq.}} = \frac{1}{3}\mathcal{E}$$

$$W_0^{a} = \frac{2}{(2\pi)^3\sqrt{2\pi\sigma}} \mathrm{e}^{-\frac{v^2}{2\tau_0^2\sigma}} e^{-\frac{1}{T_0}\sqrt{\sigma=k_T^2+\frac{w^2}{\tau_0^2}}} [1 - 3P_2\left(\frac{w}{\tau_0\sqrt{\sigma}}\right)] \longrightarrow \mathcal{P}_T^0 = \frac{8}{5}\mathcal{P}_{\mathrm{eq.}}, \quad \mathcal{P}_L^0 = -\frac{1}{5}\mathcal{P}_{\mathrm{eq.}}$$

10.1103/PhysRevD.108.036015

Hydrodynamics

$$\dot{\mathcal{E}} = -\frac{\mathcal{E}+\mathcal{P}_L}{\tau}$$

$$\dot{\mathcal{P}}_L + \frac{1}{\tau_R}\left(\mathcal{P}_L - \frac{1}{3}\mathcal{E}\right) = -\frac{3}{\tau}\mathcal{P}_L + \frac{1}{\tau}\mathcal{R}_L^{(1)}\Big|_{eq}$$

$$\dot{\mathcal{P}}_T + \frac{1}{\tau_R}\left(\mathcal{P}_T - \frac{1}{3}\mathcal{E}\right) = -\frac{1}{\tau}\mathcal{P}_T + \frac{1}{\tau}\mathcal{R}_T^{(1)}\Big|_{eq}$$

What can we say for the isotropic case

$$\mathrm{R}_L^{eq.} = \frac{1}{5}\mathcal{E} \qquad \mathrm{R}_L^{0} = -\frac{1}{5}\mathcal{E}$$

$$\mathrm{R}_T^{eq.} = \frac{1}{15}\mathcal{E} \qquad \mathrm{R}_T^{0} = -\frac{1}{15}\mathcal{E}$$

$$\frac{\delta\dot{\mathcal{P}}_L}{\dot{\mathcal{P}}_L}\Big|_0 = -\frac{1}{3}$$

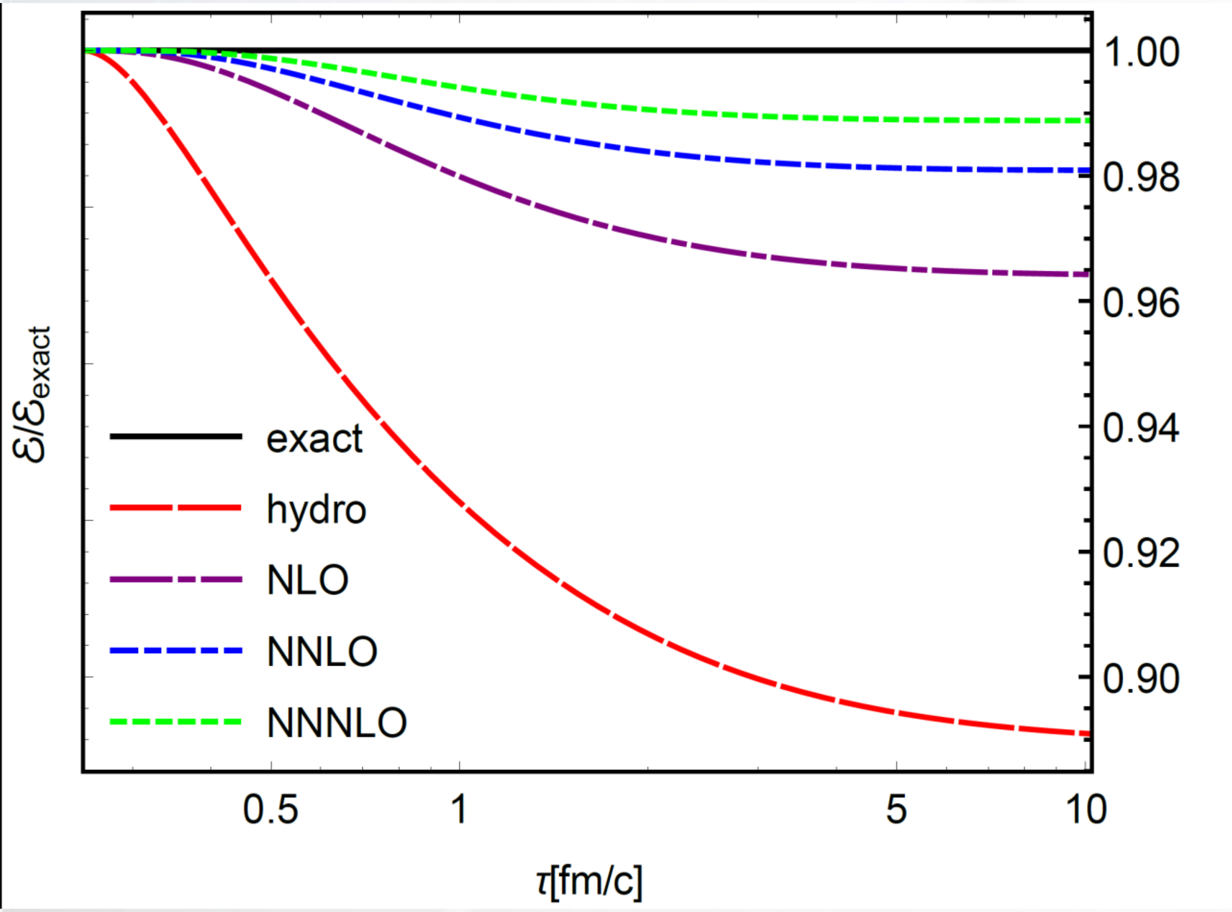
$$\frac{\delta\dot{\mathcal{P}}_T}{\dot{\mathcal{P}}_T}\Big|_0 = -\frac{1}{3}$$

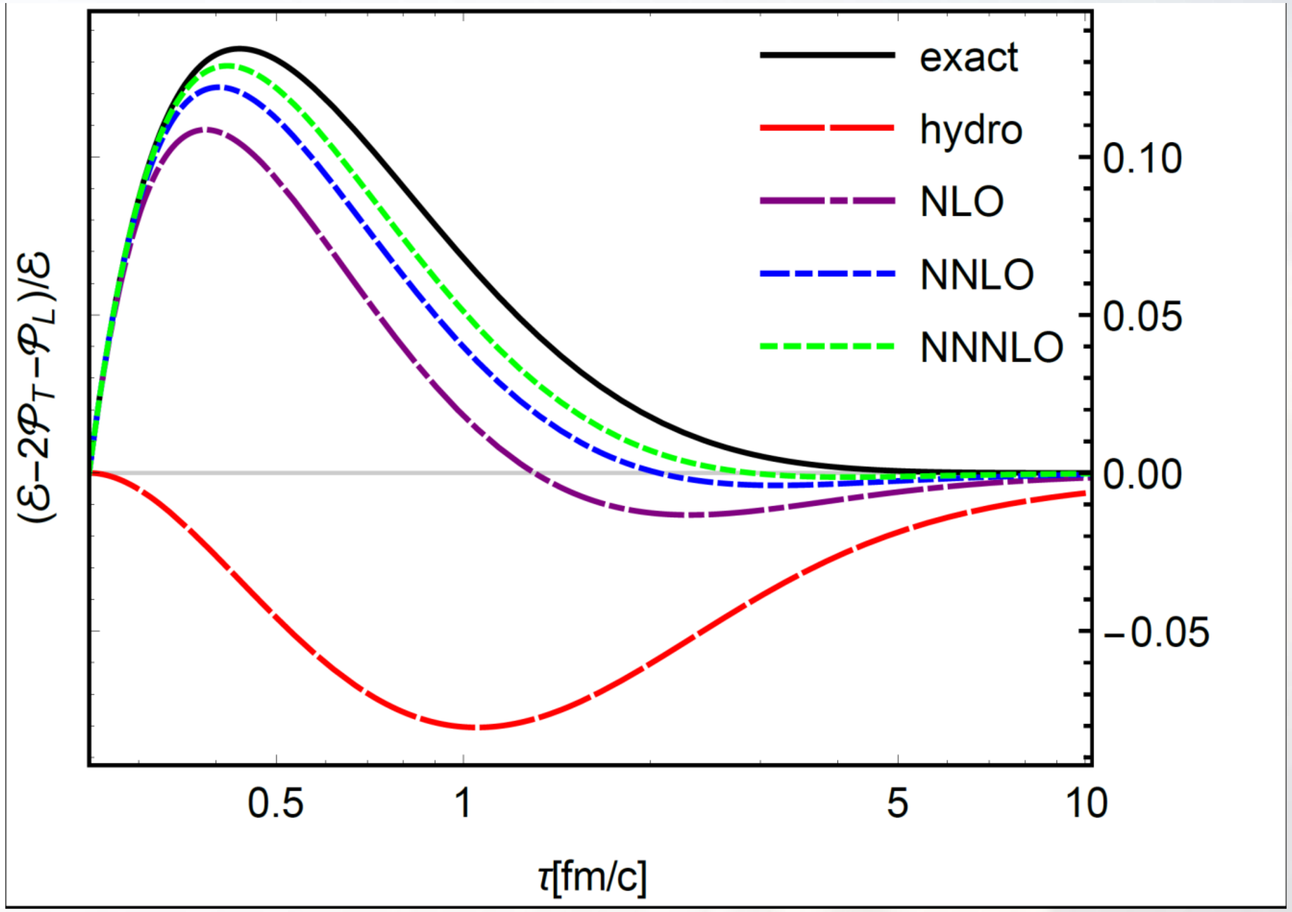
$$\delta\mathcal{P}_L = \int_{\tau_0}^{\tau} ds\, \delta\dot{\mathcal{P}}_L \Rightarrow \frac{\delta\mathcal{P}_L}{\mathcal{P}_L} = \frac{\int \delta\dot{\mathcal{P}}_L}{\mathcal{P}_L} \Rightarrow \text{Maximum if } 0 = \partial_\tau\left(\frac{\delta\mathcal{P}_L}{\mathcal{P}_L}\right) = \frac{\delta\dot{\mathcal{P}}_L}{\mathcal{P}_L} - \frac{\delta\mathcal{P}_L}{\mathcal{P}_L}\frac{\dot{\mathcal{P}}_L}{\mathcal{P}_L} \Rightarrow \frac{\delta\mathcal{P}_L}{\mathcal{P}_L} = \frac{\delta\dot{\mathcal{P}}_L}{\dot{\mathcal{P}}_L}$$

$$\frac{\delta\mathcal{E}}{\mathcal{E}} = \frac{\delta\dot{\mathcal{E}}}{\dot{\mathcal{E}}} = \frac{\delta\mathcal{E}+\delta\mathcal{P}_L}{\mathcal{E}+\mathcal{P}_L} \Rightarrow \frac{\delta\mathcal{E}}{\mathcal{E}} \simeq \frac{\delta\mathcal{P}_L}{\mathcal{P}_L}$$

…but for the trace anomaly $\mathcal{E} - 2\mathcal{P}_T - \mathcal{P}_L = -3\Pi$ $\qquad \frac{\delta\dot{\Pi}}{\dot{\Pi}} = -1$

Comparisons with the exact solutions

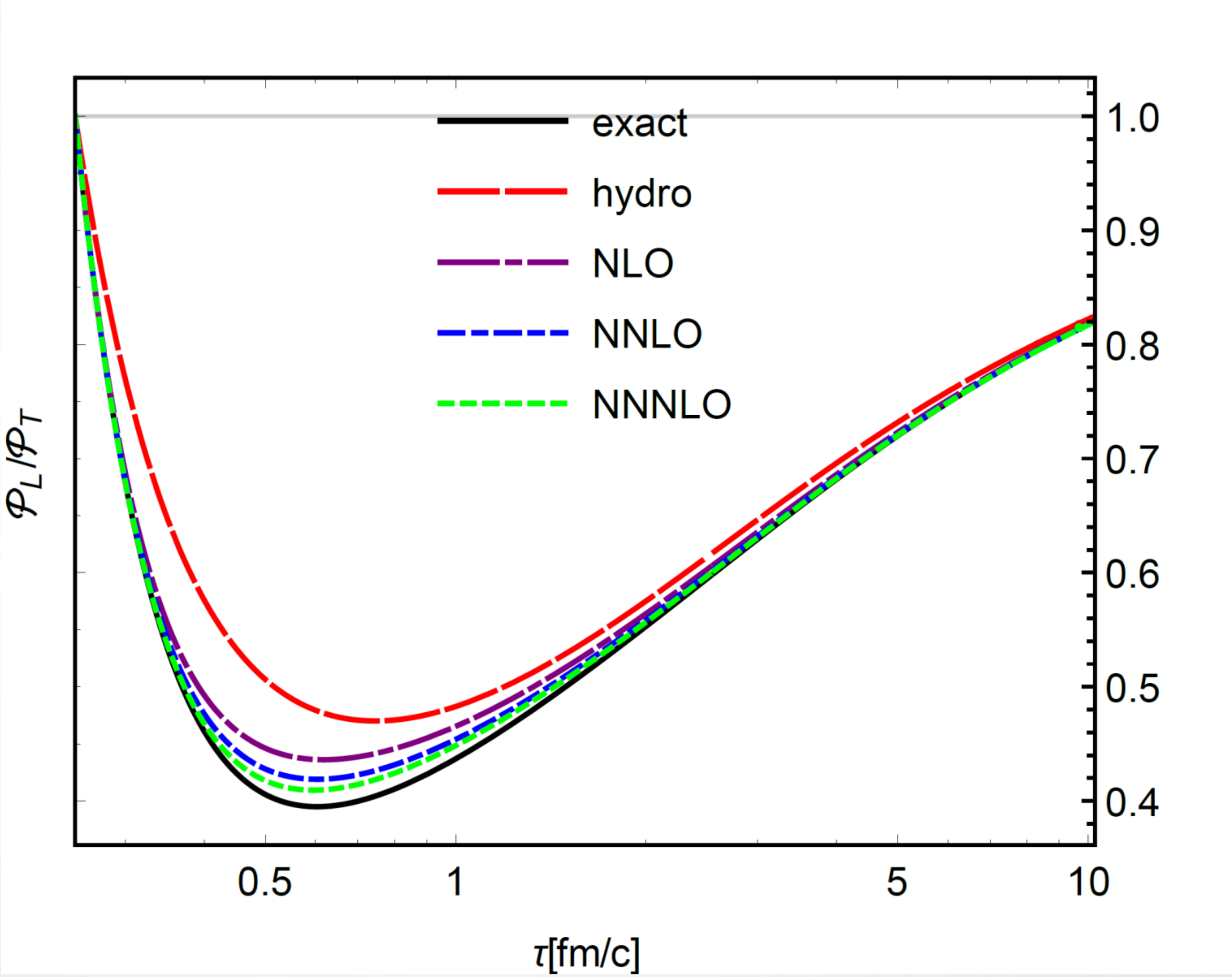




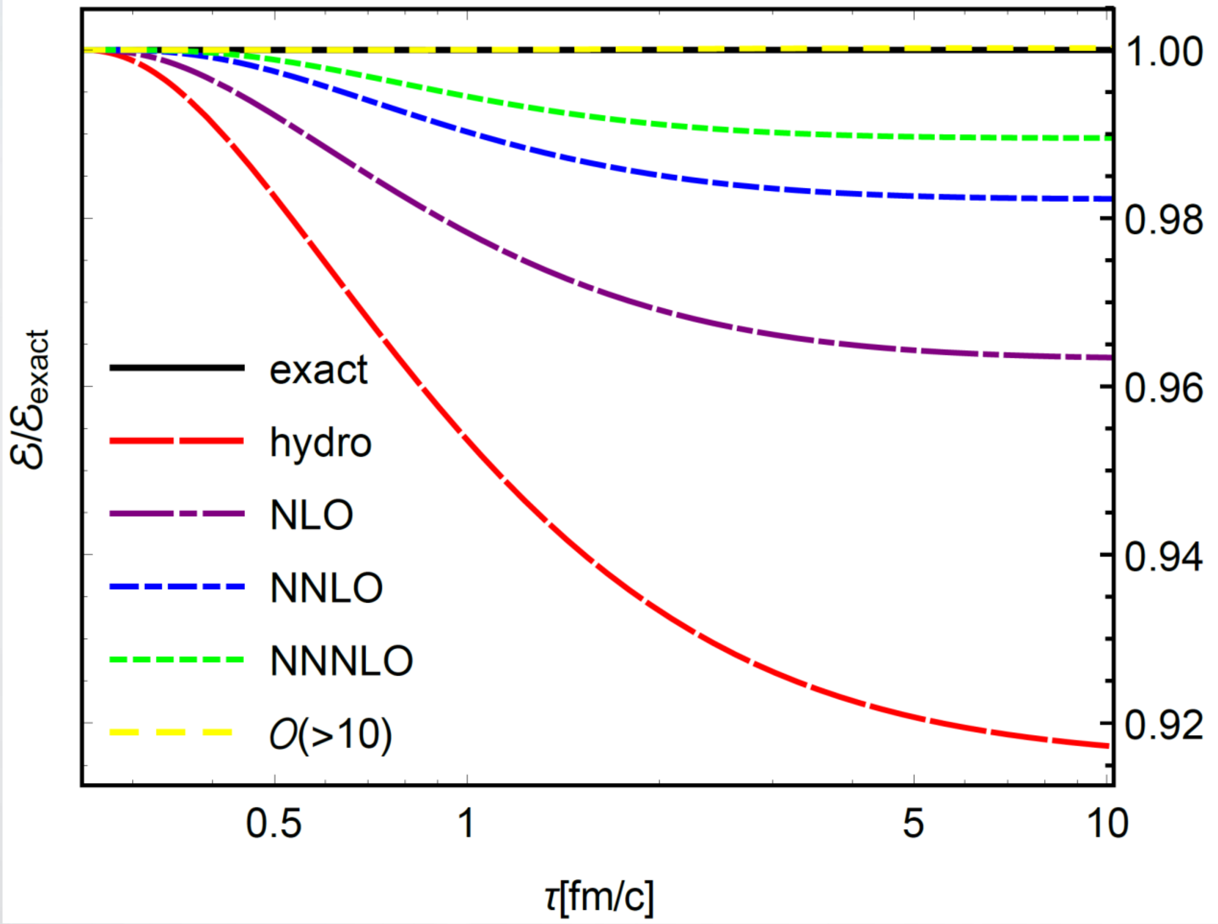
$$(\mathcal{E} - 2\mathcal{P}_T - \mathcal{P}_L)/\mathcal{E} = -\frac{3\Pi}{\mathcal{E}} = -\frac{\Pi}{\mathcal{P}}$$

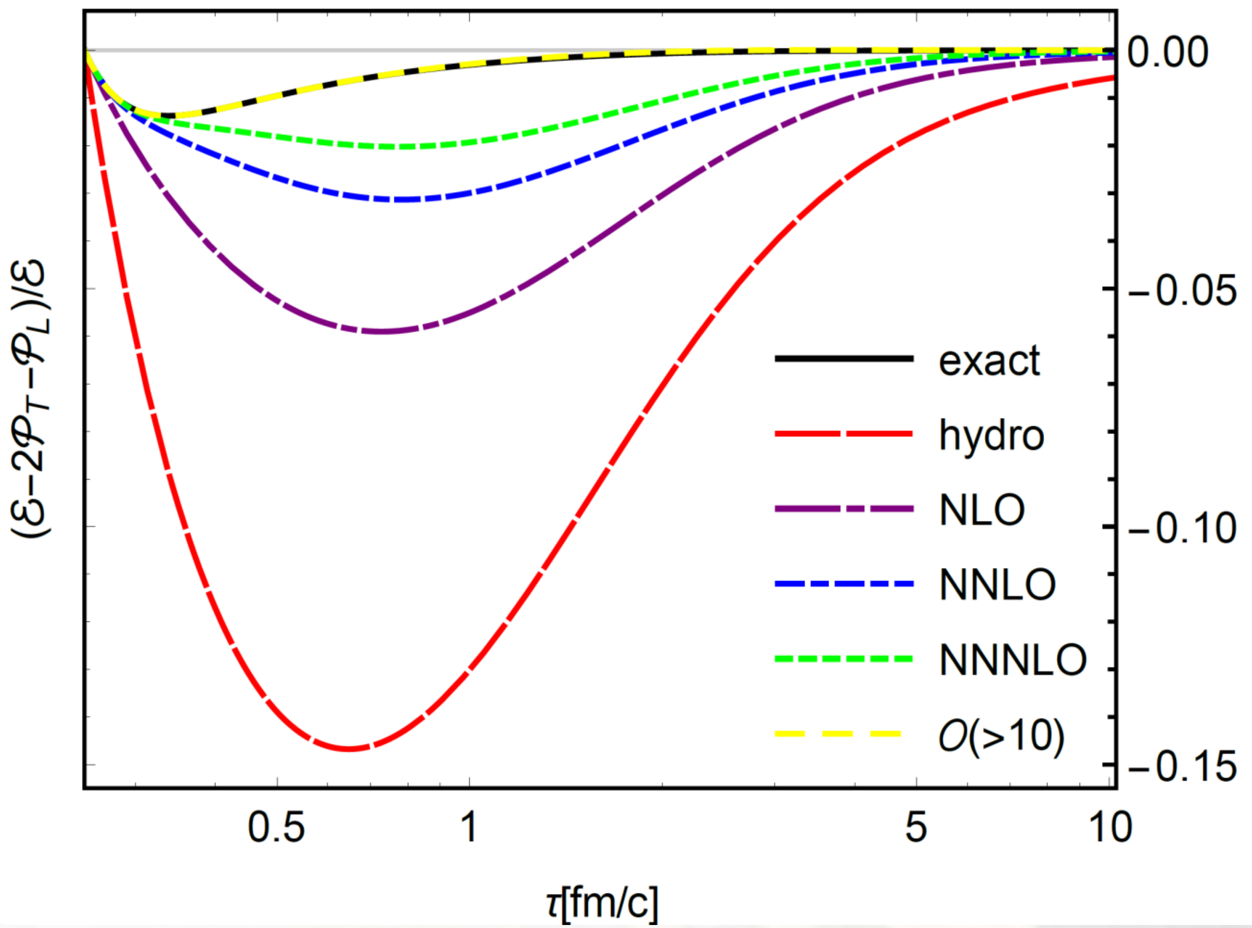
Comparisons with the exact solutions

fast convergence for the pressure anisotropy too



Comparisons for the anisotropic initial conditions

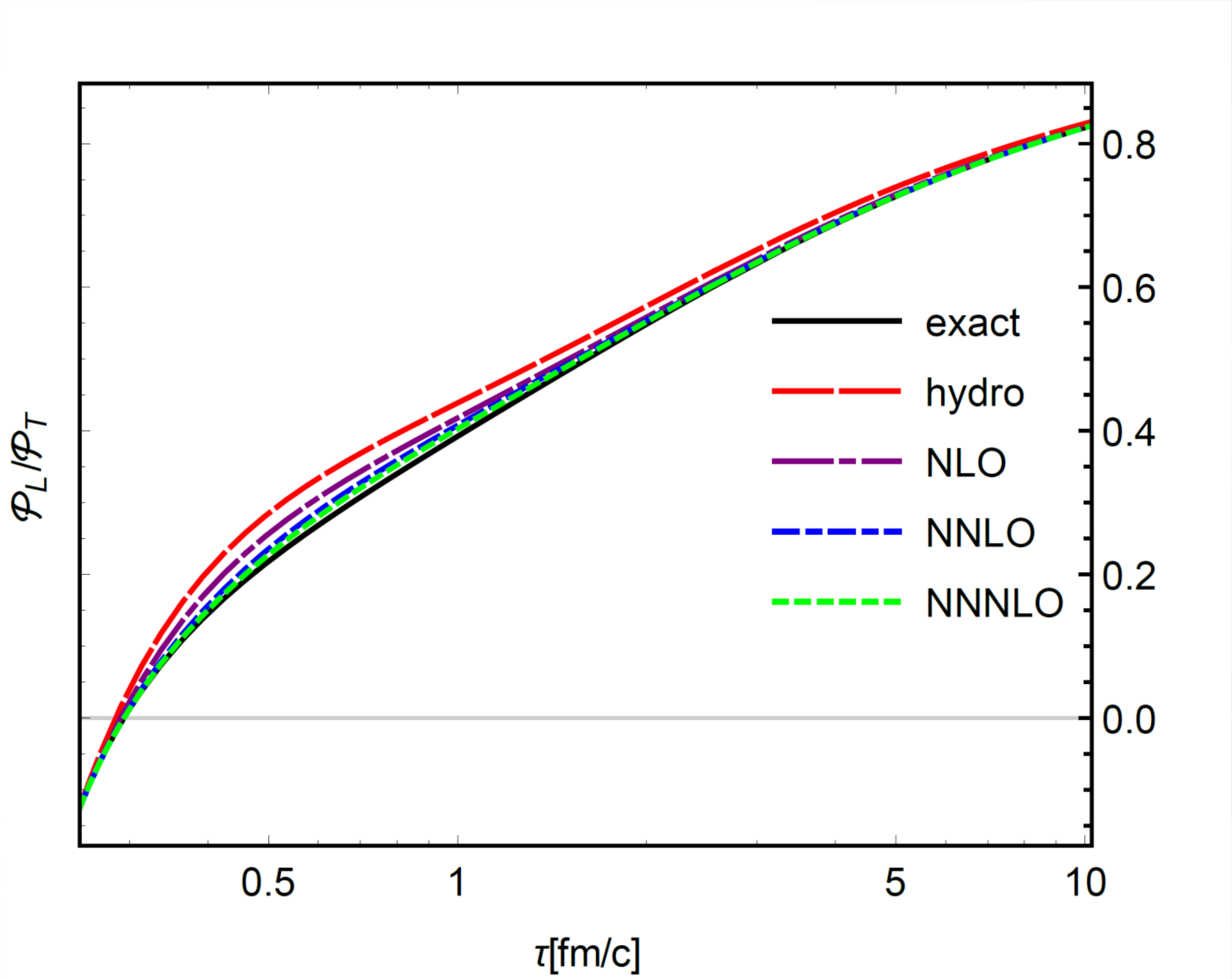




similar conclusions

Comparisons for the anisotropic initial conditions

reasonable approximation for the pressure anisotropy from the start



similar conclusions

Non-trivial vacuum behavior

From a discussion between Sinyukov and Akkelin regarding the statistical non-equilibrium density operator

$$\hat{\rho}_{NEDO} = \frac{e^{-\hat{\Upsilon}}}{Z_\Upsilon} = \frac{1}{Z_\Upsilon} \exp\left\{\int d\Sigma_\mu \frac{\hat{T}^{\mu\nu} u_\nu}{T}\right\}$$

The proper way to renormalize the operators to get the expectation values starts from removing the minimum of the $\hat{\Upsilon}$ operator, not the Hamiltonian (Minkowski vacuum)

But the particle number operator (and momentum etc.) is simply normal ordered:

The ``Υ vacuum '' contributes to the total number of particles. (similar to the Unrhu vacuum contribution)

S V Akkelin, 10.1140/epja/i2019-12755-9, 10.1103/PhysRevD.103.116014

Exactly solvable case:

quantum scalar with a classical source

The field equations from the Lagrangian density can be solved exactly

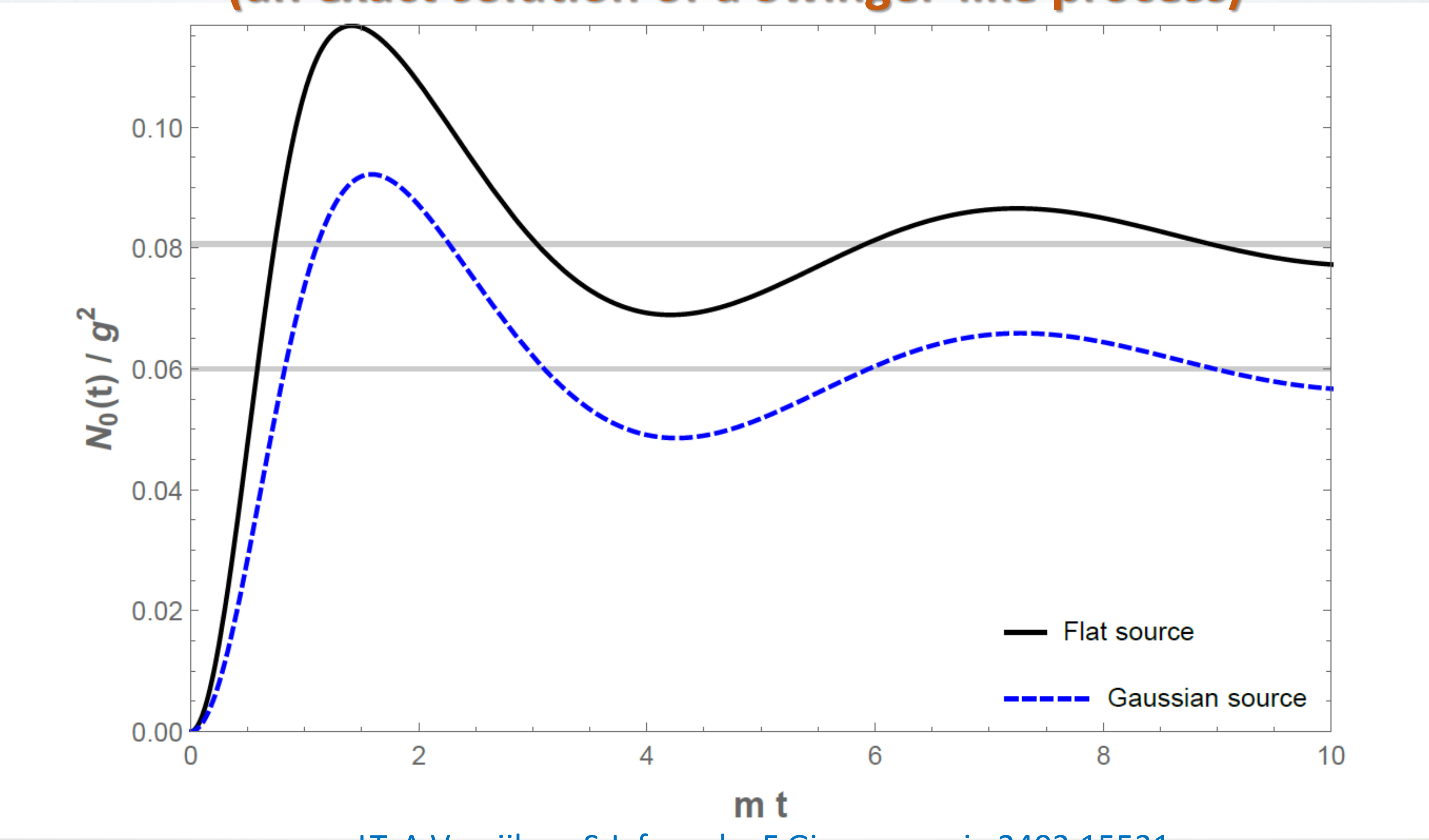
$$\frac{1}{2}\partial_\mu \hat{\phi}\, \partial^\mu \hat{\phi} - \frac{1}{2} m^2 \hat{\phi}^2 + g\, \rho\, \hat{\phi}$$

$$\hat{\phi}(x) = \hat{\phi}_0(x) + ig \int_{t_0}^{t} dy^0 \int d^3y\, \rho(y) \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right)$$

One can solve exactly for the evolution of any state (LT, A Vereijken, S Jafarzade, F Giacosa, arxiv:2403.15531)**, the evolution is an interference pattern between the vacuum decay, the initial state (that evolves as a free state) and the absorption amplitudes of the initial particles.**

Particular case: vacuum decay

(an exact solution of a Swinger-like process)



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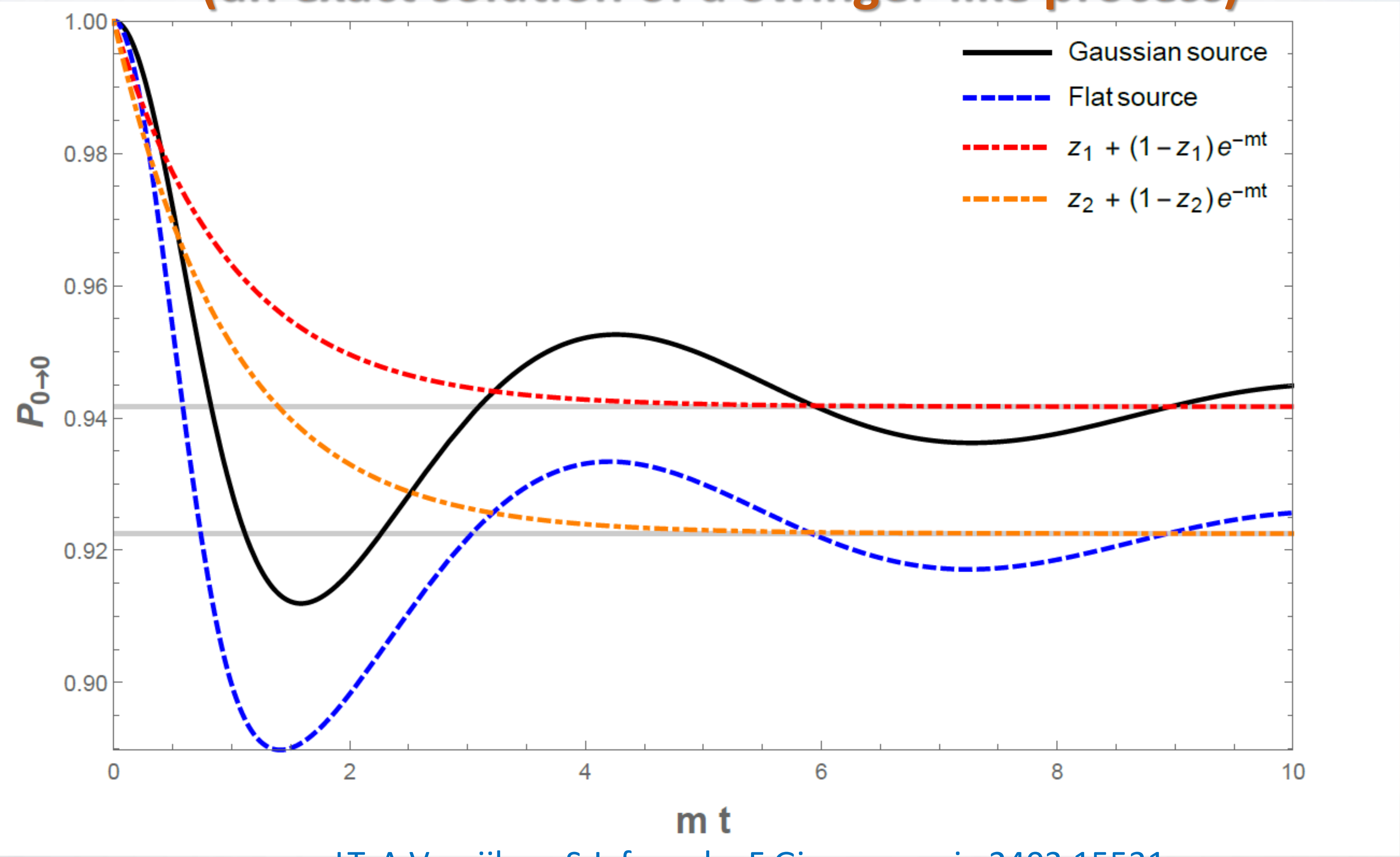


LT, A Vereijken, S Jafarzade, F Giacosa, arxiv:2403.15531

Conclusions and outlook

- Large quantum corrections (especially initial stages)
- Qualitative behavior similar, off-shell generalization (eg transport coefficients)
- Non trivial Vacuum decay as a source of particles

Thank you for your attention!

Back up slides

$$\int [g(x) + h(x)]\, dx \neq \int g(x)dx + \int h(x)dx$$

$$\int \lim_{\varepsilon\to 0} f(\varepsilon, x)dx \neq \lim_{\varepsilon\to 0} \int f(\varepsilon, x)dx$$

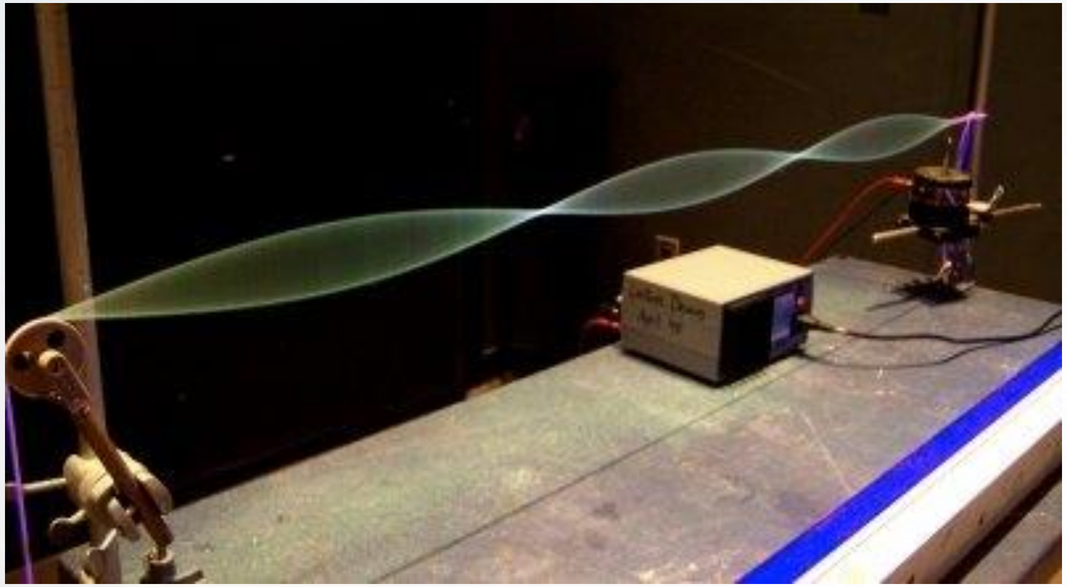
$$\frac{1}{\beta} = \int_0^\infty \left[-\partial_\beta \left(\frac{e^{-\beta x}}{x}\right)\right] dx \neq -\partial_\beta \left(\int_0^\infty \frac{e^{-\beta x}}{x} dx \equiv \infty\right)$$

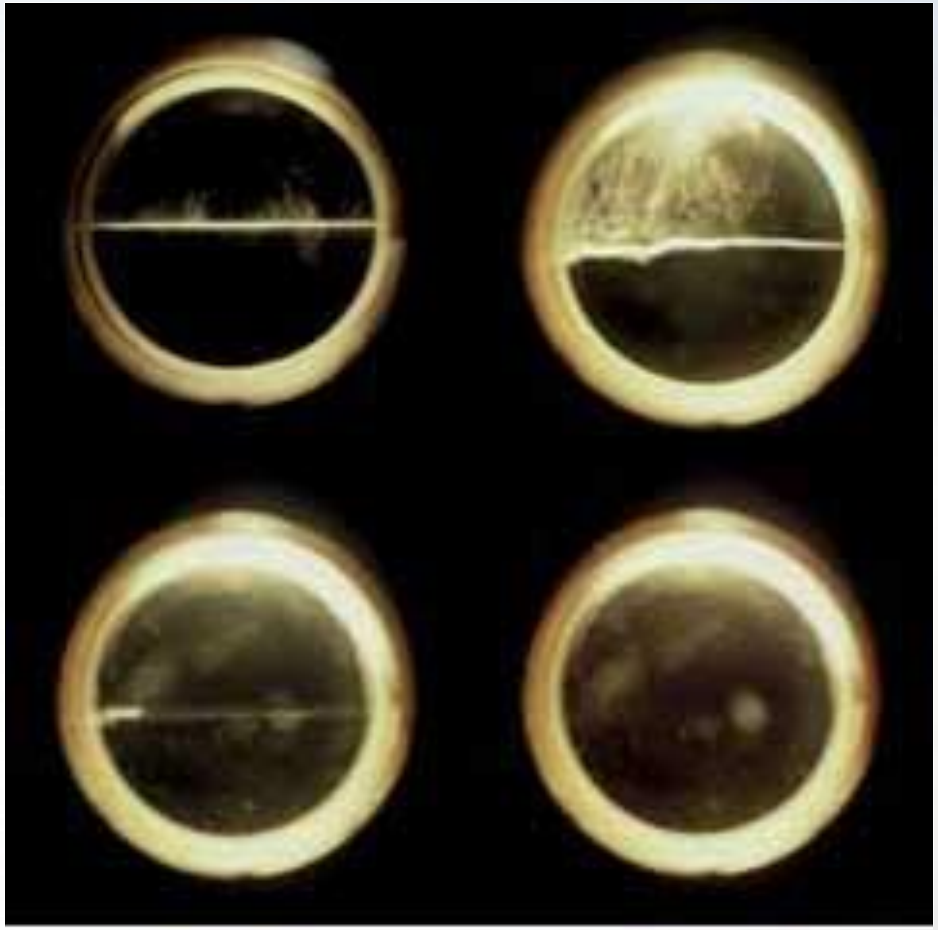
$$\frac{1}{x} = \int_0^\infty e^{-\alpha x} d\alpha$$

$$\frac{1}{(\alpha+\beta)^2} = \int_0^\infty dx \left[-\partial_\beta \left(e^{-(\alpha+\beta)x}\right)\right] = -\partial_\beta \left(\int_0^\infty dx\, e^{-(\alpha+\beta)x} = \frac{1}{\alpha+\beta}\right),$$

$$\int_0^\infty d\alpha \left[\frac{1}{(\alpha+\beta)^2} = \partial_\alpha \left(-\frac{1}{\alpha+\beta}\right)\right] = \frac{1}{\beta}$$

Hydrodynamics

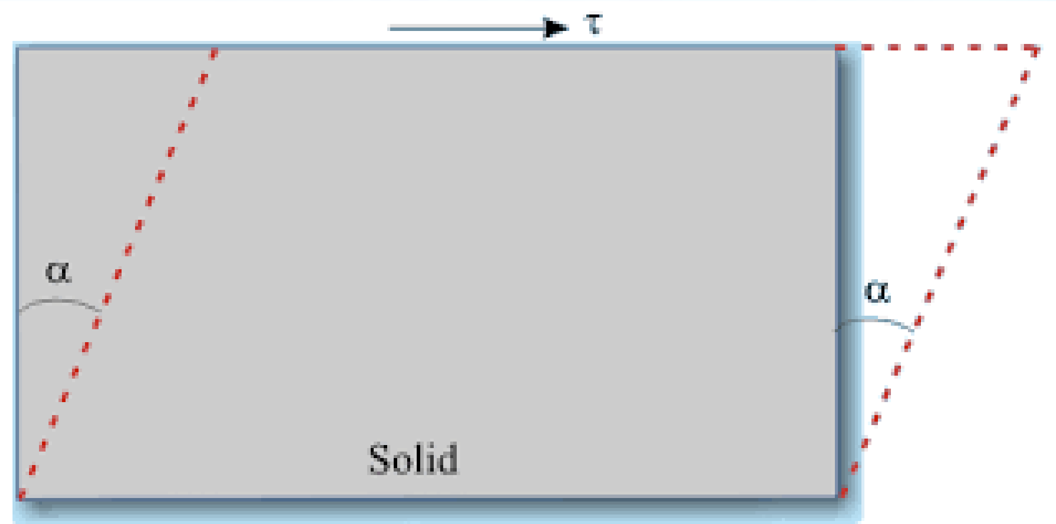




Hydrodynamics is the low-energy,
long wave-length limit of a theory

Hydrodynamics require small gradients/deviations from equilibrium

Hydrodynamics



$$\partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho\, \boldsymbol{v}) = 0$$

$$\rho \boldsymbol{a} = -\nabla_{\mathbf{x}} \mathrm{P}$$

generalizes to

$$\rho \boldsymbol{a} = \nabla_x \cdot T$$

<u>the Cauchy Tensor T</u>

<u>definition of a fluid:</u>

$$T_{ij}\Big|_{eq} = -\mathrm{P}\,\delta_{ij}$$

For an incompressible fluid $\nabla_x \cdot \boldsymbol{v} = \mathbf{0}$

$$T_{ij} \simeq -\mathrm{P}\,\delta_{ij} + \eta\left(\partial_i v_j + \partial_j v_i\right) + \cdots$$

Relativistic hydrodynamics

$$\boldsymbol{v} \to u = \begin{pmatrix} \gamma \\ \gamma \boldsymbol{v}/c \end{pmatrix}$$
$$\rho \to \mathcal{E}$$

relativistic degrees of freedom

projector: $\Delta^{\mu\nu} = u^\mu u^\nu - g^{\mu\nu}$

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu - \mathrm{P}\Delta^{\mu\nu} + \cdots$$

$$\partial_\mu \mathrm{T}^{\mu\nu} = 0$$

local four-momentum conservation

which implies

$$\begin{cases} 0 = u_\nu \partial_\mu T^{\mu\nu} \xrightarrow{c\to\infty} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho\, \boldsymbol{v}) \\ 0 = \Delta_{i\nu} \partial_\mu T^{\mu\nu} \xrightarrow{c\to\infty} (\rho \boldsymbol{a} + \nabla_{\mathbf{x}} \mathrm{P})\Big|_i + \cdots \end{cases}$$

Relativistic hydrodynamics

$$\left.\begin{aligned} \partial_\mu \hat{T}^{\mu\nu} &= 0 \\ T^{\mu\nu} &= tr(\hat{\rho}\,\hat{T}^{\mu\nu}) \end{aligned}\right\} \rightarrow \partial_\mu T^{\mu\nu} = 0$$

Hydro

$$T^{\mu\nu} = \mathcal{E}\, u^\mu\, u^\nu - \mathcal{P}\Delta^{\mu\nu} + \delta T^{\mu\nu}$$

From quantum field theory, but at least ten degrees of freedom and only four equations

Gradient expansion

- Requires small gradients
- Unstable (even in the non-relativistic limit)
- Not converging

A Buchel, M P Heller, J Noronha, arXiv:1603.05344

G Denicol, J Noronha, arXiv:1608.07869

$$\delta T^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \cdots$$

↓

transport coefficients times gradients

Simplest case: free streaming

Classical limit of the exact solutions

$$\lim_{\hbar\to 0}\left[(2\pi\hbar)^3\, W(x,k)\right] \propto \delta(k^2 - m^2)$$

$$\chi = 2\sqrt{\frac{k^2 - m^2}{k^2 - m_T^2}}$$

Particles

(similar for the antiparticles)

$$(2\pi\hbar)^3\, W^+ = \theta(k^0)\theta(k^2 - m^2c^2)\frac{(4-\chi^2)^2}{4m_T^2\chi}\left[\cos\left(\frac{w\chi}{\hbar}\right)\tilde{f}_{\text{even}}(k^0, k_T, k^z) + \sin\left(\frac{w\chi}{\hbar}\right)\tilde{f}_{\text{odd}}(k^0, k_T, k^z)\right]\frac{A}{2\pi\hbar}$$

$$\varepsilon = \frac{\hbar}{A}$$

$$\int \frac{dx}{\varepsilon}\, g\left(\frac{x}{\varepsilon}; x, p_1\cdots\right)\psi(x) = \int dy\; g\left(y; y\varepsilon, p_1\cdots\right)\psi(y\varepsilon) \xrightarrow{\varepsilon\to 0} \psi(0)\int dy\; g(y; 0, p_1, \cdots),$$

$$\Rightarrow \qquad \frac{1}{\varepsilon}\, g\left(\frac{x}{\varepsilon}; x, p_1\cdots\right) \xrightarrow{\varepsilon\to 0} \delta(x)\int dy\; g(y; 0, p_1, \cdots).$$

10.1103/PhysRevD.108.076022

Simplest case: free streaming

Classical limit of the exact solutions

$$\varepsilon = \frac{\hbar}{A} \qquad \tilde{w} = \frac{w}{A}$$

$$\frac{\theta(\chi)\theta(2-\chi)}{(2\pi\varepsilon)}\cos\left(\tilde{w}\frac{\chi}{\varepsilon}\right)\ \tilde{f}_{\rm even}\left(\frac{\chi}{\varepsilon};k^0,k_T,k^z\right)\xrightarrow{\varepsilon\to 0^+}\frac{1}{2}\delta(\chi)\int\frac{d\chi'}{(2\pi)}\cos\left(\tilde{w}\chi'\right)\tilde{f}_{\rm even}\left(\chi';\sqrt{m_T^2+(k^z)^2},k_T,k^z\right)$$

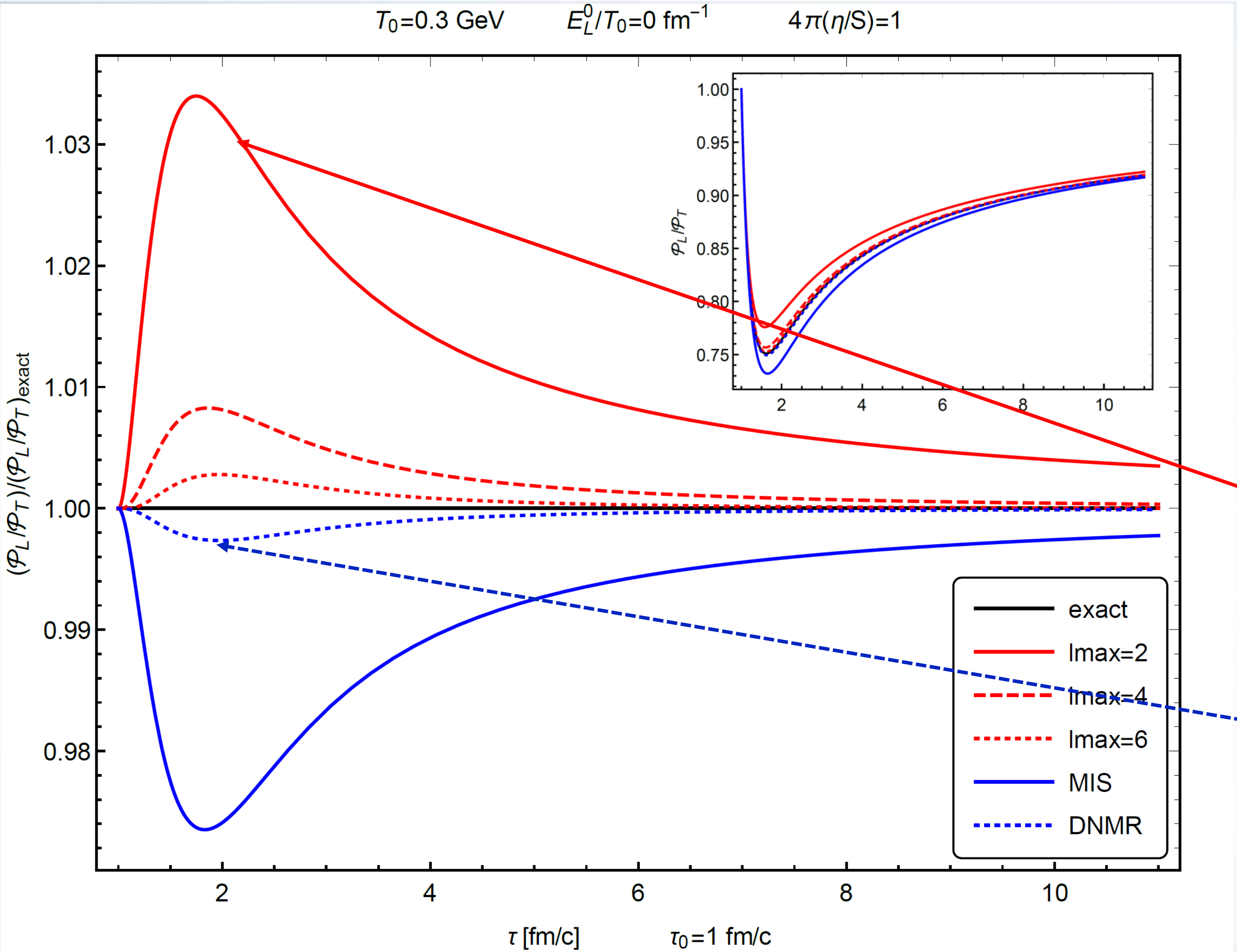
$$\frac{\theta(\chi)\theta(2-\chi)}{(2\pi\varepsilon)}\sin\left(\tilde{w}\frac{\chi}{\varepsilon}\right)\ \tilde{f}_{\rm odd}\left(\frac{\chi}{\varepsilon};k^0,k_T,k^z\right)\xrightarrow{\varepsilon\to 0^+}\frac{1}{2}\delta(\chi)\int\frac{d\chi'}{(2\pi)}\sin\left(\tilde{w}\chi'\right)\tilde{f}_{\rm odd}\left(\chi';\sqrt{m_T^2+(k^z)^2},k_T,k^z\right)$$

Proportional to the real (hence even in $\tilde{w}$) and imaginary (odd) part of the Fourier transform

$$\tilde{f}_{\rm even}\left(\frac{\chi}{\varepsilon};k_T,k^z\right)=2{\rm Re}\left[\int d\tilde{w}'\ f\left(\tilde{w}';k_T,k^z\right)\ e^{-i\tilde{w}'\frac{\chi}{\varepsilon}}\right]$$

$$\tilde{f}_{\rm odd}\left(\frac{\chi}{\varepsilon};k_T,k^z\right)=2{\rm Im}\left[\int d\tilde{w}'\ f\left(\tilde{w}';k_T,k^z\right)\ e^{-i\tilde{w}'\frac{\chi}{\varepsilon}}\right]$$

10.1103/PhysRevD.108.076022



The method of moments converges fast

different treatment of the residual moments

$$\rho_{-2}^{\mu_1\cdots\mu_4} \to \rho_{-2}^{\mu_1\cdots\mu_4}\Big|_{eq.}$$

$$\rho_{-1}^{\mu_1\cdots\mu_4} \neq \rho_{-2}^{\mu_1\cdots\mu_4}\Big|_{eq.}$$

L T, G Vujnovich, J Noronha, U Heinz **arXiv:1808.06436**

L T, G Vujnovich **(WIP)**

Particles interacting with external fields

Boltzmann-Vlasov equation

$$p \cdot \partial f + m\partial_\alpha m\, \partial^\alpha_{(p)} f + qF_{\alpha\beta}p^\beta \partial^\alpha_{(p)} f = -\mathcal{C}[f]$$

Immediate (but problematic) generalization

$$\begin{aligned}\dot{\mathcal{F}}_r^{\mu_1\cdots\mu_s} + C_{r-1}^{\mu_1\cdots\mu_s} \;=\;& r\,\dot{u}_\alpha\,\mathcal{F}_{r-1}^{\alpha\mu_1\cdots\mu_s} - \nabla_\alpha \mathcal{F}_{r-1}^{\alpha\mu_1\cdots\mu_s} + (r-1)\,\nabla_\alpha u_\beta\,\mathcal{F}_{r-2}^{\alpha\beta\mu_1\cdots\mu_s} \\ & + m\dot{m}\,(r-1)\,\mathcal{F}_{r-2}^{\mu_1\cdots\mu_s} + s\,m\partial^{(\mu_1} m\,\mathcal{F}_{r-1}^{\mu_2\cdots\mu_s)} \\ & - q(r-1)\,E_\alpha\,\mathcal{F}_{r-2}^{\alpha\mu_1\cdots\mu_s} - q\,s\,g_{\alpha\beta}F^{\alpha(\mu_1}\mathcal{F}_{r-1}^{\mu_2\cdots\mu_s)\beta}\end{aligned}$$

$$F_{\mu\nu} = E_\mu u_\nu - E_\nu u_\mu + \varepsilon_{\mu\nu\rho\sigma} u^\rho B^\sigma$$

Moments with large negative r needed, infrared catastrophe!

arXiv:1808.06436

Exactly solvable case

Bjorken symmetry

$$\tau = \sqrt{t^2 - z^2}, \qquad v = k^0 t - z\, k^z, \quad u = (\cosh\eta, 0,0, \sinh\eta)$$

$$\eta = \frac{1}{2}\ln\left(\frac{t+z}{t-z}\right), \qquad w = z k^0 - t\, k^z, \quad z = (\sinh\eta, 0,0, \cosh\eta)$$

(as a consequence)

$$T^{\mu\nu} = \mathcal{E}(\tau) u^\mu u^\nu + \mathcal{P}_T(\tau)\,(x^\mu x^\nu + y^\mu y^\nu) + \mathcal{P}_L(\tau) z^\mu z^\nu$$

$$\pi^{\mu\nu} = -\frac{1}{2}\pi(\tau)\,(x^\mu x^\nu + y^\mu y^\nu) + \pi(\tau) z^\mu z^\nu$$

$$\mathcal{P}_T = \mathcal{P} + \Pi - \frac{1}{2}\pi, \qquad \mathcal{P}_L = \mathcal{P} + \Pi + \pi$$

RTA

$$k\cdot\partial W = -\frac{k\cdot u}{\tau_R}(W - W_{eq}) = -\frac{k\cdot u}{\tau_R}\left(W - \frac{2\delta(k^2)}{(2\pi)^3} e^{-\frac{1}{T(\tau)}\sqrt{k_T^2 + \frac{w^2}{\tau^2}}}\right) \Rightarrow \partial_\tau W + 2\frac{v^2 - w^2}{\tau}\partial v^2 W = \frac{1}{\tau_R}\delta W$$

in addition $W(\tau, v^2, k_T, w^2)$

10.1103/PhysRevD.108.036015

Resummed moments

Making use of regularized moments

$$\phi_n^{\mu_1\cdots\mu_s}(x,\zeta) = \int \frac{d^4k}{(2\pi)^4}\,(k\cdot u)^n\, e^{-\zeta(k\cdot u)^2} k^{\langle\mu_1\rangle}\ldots k^{\langle\mu_s\rangle} W(x,k)$$

$$\partial_\zeta \phi_n^{\mu_1\cdots\mu_s} = -\phi_{n+2}^{\mu_1\cdots\mu_s}$$

$$\int_\zeta^\infty dv\, \phi_{n+2}^{\mu_1\cdots\mu_s} = \phi_n^{\mu_1\cdots\mu_s}$$

$$\phi_n^{\mu_1\cdots\mu_s}(x,0) = \Delta_{\boldsymbol{\alpha_1}}^{\boldsymbol{\mu_1}}\cdots\Delta_{\boldsymbol{\alpha_s}}^{\boldsymbol{\mu_s}}\mathcal{F}_{\mathrm{n}}^{\alpha_1\cdots\alpha_s} = \mathfrak{f}_n^{\alpha_1\cdots\alpha_s}$$

All (well-defined) previous moments recovered from the resumed ones, including $T^{\mu\nu}$

2 generations of dynamical moments needed

$$\dot\phi_2^{\langle\mu_1\rangle\cdots\langle\mu_1\rangle} + \tilde C_1^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} = -\theta\phi_2^{\mu_1\cdots\mu_s} - s\nabla_\alpha u^{(\mu_1}\phi_2^{\mu_2\cdots\mu_s)\alpha} - \nabla_\alpha\phi_1^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + \dot u_\alpha\left[2\phi_1^{\alpha\mu_1\cdots\mu_s} + 2\zeta\partial_\zeta\phi_1^{\alpha\mu_1\cdots\mu_s}\right]$$
$$-s\dot u^{(\mu_1}\partial_\zeta\phi_1^{\mu_2\cdots\mu_s)} + \nabla_\alpha u_\beta\left[\int_\zeta^\infty dv\,\phi_2^{\alpha\mu_1\cdots\mu_s} - 2\zeta\phi_2^{\alpha\mu_1\cdots\mu_s}\right]$$

$$\dot\phi_1^{\langle\mu_1\rangle\cdots\langle\mu_1\rangle} + \tilde C_0^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} = -\theta\phi_1^{\mu_1\cdots\mu_s} - s\nabla_\alpha u^{(\mu_1}\phi_1^{\mu_2\cdots\mu_s)\alpha} - \nabla_\alpha\int_\zeta^\infty dv\phi_1^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + \dot u_\alpha\left[\int_\zeta^\infty dv\,\phi_2^{\alpha\mu_1\cdots\mu_s} - 2\zeta\phi_2^{\alpha\mu_1\cdots\mu_s}\right]$$
$$+s\dot u^{(\mu_1}\phi_2^{\mu_2\cdots\mu_s)} - 2\zeta\nabla_\alpha u_\beta\phi_1^{\alpha\beta\mu_1\cdots\mu_s}$$

10.1103/PhysRevD.108.036015

Hydrodynamic expansion

$$L_n = \phi_2^{\mu_1\cdots\mu_{2n}} z_{\mu_1}\cdots z_{\mu_{2n}}, \qquad T_n = \phi_2^{\mu_1\cdots\mu_{2n}\alpha\beta} z_{\mu_1}\cdots z_{\mu_{2n}} x_\alpha x_\beta$$

$$\dot{L}_n + \frac{1}{\tau_R}(L_n - L_n^{eq.}) = -\frac{2n+1}{\tau} L_n + \frac{1}{\tau}\hat{\mathcal{L}} L_{n+1}$$

$$\dot{T}_n + \frac{1}{\tau_R}(T_n - T_n^{eq.}) = -\frac{2n+1}{\tau} T_n + \frac{1}{\tau}\hat{\mathcal{L}} T_{n+1}$$

$$\mathcal{E} = L_0(\tau, \zeta = 0)$$

$$\mathcal{P}_L = \int_\zeta^\infty d\zeta' \, L_1(\tau, \zeta')$$

$$\mathcal{P}_T = \int_\zeta^\infty d\zeta' \, T_0(\tau, \zeta')$$

$$\hat{\mathcal{L}}\,[f] = 2\zeta\, f(\zeta) - \int_\zeta^\infty d\zeta' f(\zeta')$$

Hydrodynamics

$$\dot{\mathcal{E}} = -\frac{\mathcal{E} + \mathcal{P}_L}{\tau}$$

$$\dot{\mathcal{P}}_L + \frac{1}{\tau_R}\left(\mathcal{P}_L - \frac{1}{3}\mathcal{E}\right) = -\frac{3}{\tau}\mathcal{P}_L + \frac{1}{\tau}\mathcal{R}_L^{(1)}$$

$$\dot{\mathcal{P}}_T + \frac{1}{\tau_R}\left(\mathcal{P}_T - \frac{1}{3}\mathcal{E}\right) = -\frac{1}{\tau}\mathcal{P}_L + \frac{1}{\tau}\mathcal{R}_T^{(1)}$$

one can integrate the equations in ζ

…the same for the sources and their equations…

10.1103/PhysRevD.108.036015