Microscopic Encoding of Macroscopic Universality: Scaling Properties of Dirac Eigenspectra near QCD Chiral Phase Transition

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Chiral transition and universal *O(N)* scaling of the QCD transition

Microscopic origin of universal scaling near the chiral transition

H.T. Ding, W.-P. Huang, S. Mukherjee, PP, PRL 131 (2023) 161903

How criticality at macroscale arises from microscopic d.o.f. of QCD ?

$$\not{D}[\mathcal{U}]v_k = i\lambda_k v_k$$

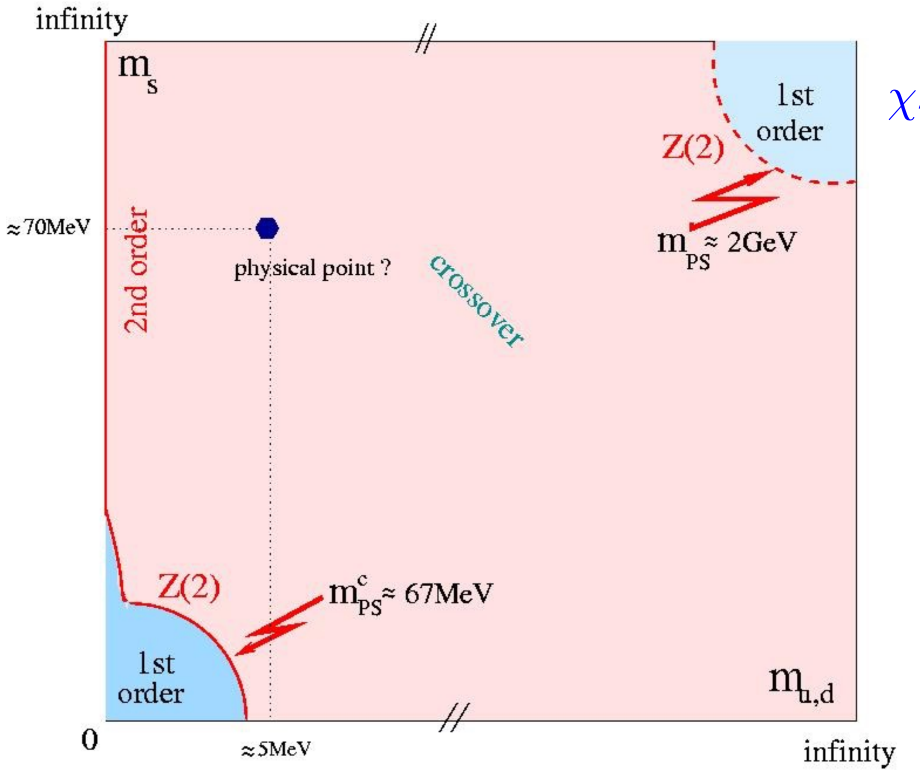
Chiral transition for small quark masses and universal scaling

$$SU(2)_V \otimes SU(2)_A \sim O(4)$$

For sufficiently small m_l the chiral transition is governed by universal *O(4)* scaling

$$f(T,m_l) = -\frac{T}{V}\ln Z = f_{reg}(T,m_l) + f_s(t,h) \qquad t = \frac{1}{t_0}\frac{T-T_c}{T_c},\ H = \frac{m_l}{m_s},\ \ h = \frac{H}{h_0}$$

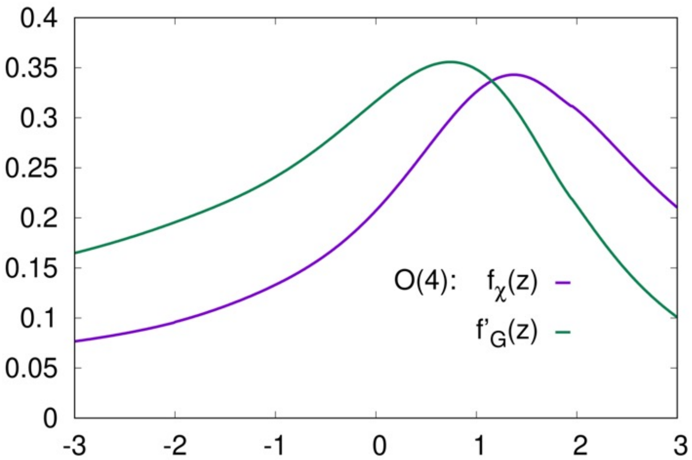
$$\langle\bar\psi\psi\rangle = T(\partial \ln Z)/\partial m_l \qquad M = -\frac{\partial f_s(t,h)}{\partial H} = h^{1/\delta} f_G(z),\ \ z = t/h^{1/\beta\delta}$$



$$\chi_m = \frac{T}{V}\frac{\partial^2 \ln Z}{\partial m_l^2} = \frac{h^{1/\delta-1}}{h_0} f_\chi(z) \longrightarrow T_{pc}$$

$$T_c = 132^{+3}_{-6}\ \text{MeV}$$

HotQCD, PRL 123 (2019) 062002

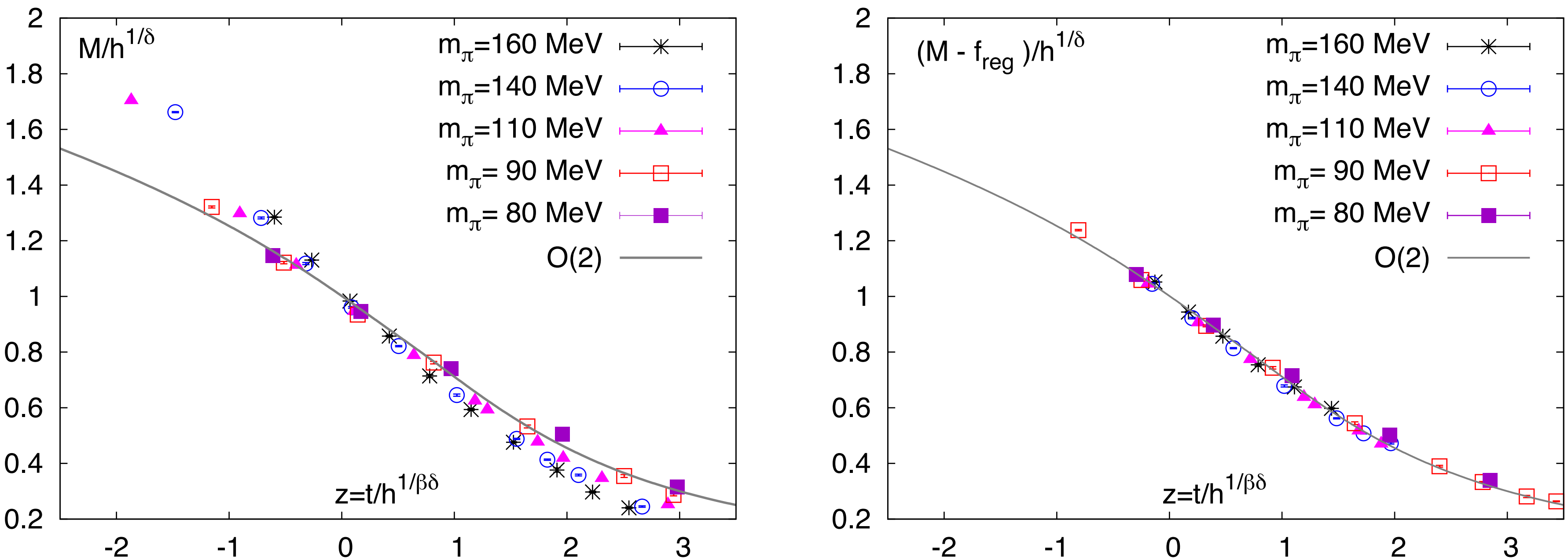


Chiral transition for small quark masses and universal scaling

HISQ action, $N_\tau = 6$ lattices

$$M = \langle\bar{\psi}\psi\rangle/T^3, M = h^{1/\delta} f_G(z) + f_{reg}(T, m_l),\ \ f_{reg} = (a_0 + a_1 \frac{T - T_c}{T_c})\frac{m_l}{m_s}$$

Staggered quarks only preserve a subgroup of chiral symmetry of QCD:
$\Rightarrow\ \ O(4) \rightarrow O(2)$



Ding et al, Lattice' 12, arXiv:1312.0119

Chiral transition and spectrum of Dirac eigenvalues

Macroscopic | Microscopic

$$\bar\psi\psi(m) \equiv 2\mathrm{Tr}(\not{D}[\mathcal{U}] + m)^{-1} \qquad = 2\sum_j (i\lambda_j + m)^{-1}$$

$$P_{\mathcal{U}}(\lambda; m) = \frac{4m\rho_{\mathcal{U}}(\lambda)}{\lambda^2 + m^2}, \text{ and } \rho_{\mathcal{U}}(\lambda) = \sum_j \delta(\lambda - \lambda_j)$$

$$\mathbb{K}_1(\bar\psi\psi) = \frac{T}{V}\langle(\bar\psi\psi)\rangle \qquad = \int_0^\infty K_1[P_{\mathcal{U}}(\lambda; m_l)]\,\mathrm{d}\lambda = \frac{T}{V}\int_0^\infty \mathrm{d}\lambda \frac{4m_l\,\langle\rho_{\mathcal{U}}(\lambda)\rangle}{\lambda^2 + m_l^2} =$$

$$= \int_0^\infty P_1(\lambda) d\lambda$$

$$\mathbb{K}_2(\bar\psi\psi) = \frac{T}{V}\langle[(\bar\psi\psi) - \langle\bar\psi\psi\rangle]^2\rangle \qquad = \int_0^\infty K_1[P_{\mathcal{U}}(\lambda_1; m_l), P_{\mathcal{U}}(\lambda_2; m_l)]\,\mathrm{d}\lambda_1\mathrm{d}\lambda_2$$

Chiral susceptibility

$$= \frac{T}{V}\int_0^\infty \mathrm{d}\lambda_1\mathrm{d}\lambda_2 \frac{(4m_l)^2}{(\lambda_1^2 + m_l^2)(\lambda_2^2 + m_l^2)} \times$$

$$[\langle\rho_{\mathcal{U}}(\lambda_1)\rho_{\mathcal{U}}(\lambda_2)\rangle - \langle\rho_{\mathcal{U}}(\lambda_1)\rangle\,\langle\rho_{\mathcal{U}}(\lambda_2)\rangle] = \int_0^\infty P_2(\lambda) d\lambda$$

$$\mathbb{K}_3(\bar\psi\psi) = \frac{T}{V}\langle[(\bar\psi\psi) - \langle\bar\psi\psi\rangle]^3\rangle \qquad = \int_0^\infty P_3(\lambda) d\lambda$$

Binder cumulant

Approaching the chiral limit

$m_l \to 0$: $\frac{m}{\lambda^2+m^2} \to \pi\delta(\lambda)$

$$\lim_{m_l\to 0} \mathbb{K}_1(\bar{\psi}\psi) = \lim_{m_l\to 0} \frac{T}{V}\langle(\bar{\psi}\psi)\rangle = 2\pi\mathbb{K}_1[\rho_{\mathcal{U}}(0)] = 2\pi\langle\rho_{\mathcal{U}}(0)\rangle$$

Banks-Casher relation

$$\lim_{m_l\to 0} \mathbb{K}_n(\bar{\psi}\psi) = \lim_{m_l\to 0} = (2\pi)^n\mathbb{K}_n[\rho_{\mathcal{U}}(0)]$$

Universal *O(N)* scaling of the cumulants of the chiral condensate:

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty P_n(\lambda)d\lambda \sim m_l^{1/\delta-n+1} f_n(z)$$

$$f_1(z) = f_G(z)$$
$$f_2(z) = f_\chi(z)$$

$$z \propto z_0 m_l^{-1/\beta\delta}(T-T_c)/T_c$$

Universal *O(N)* scaling functions

Conjecture: $P_n(\lambda) = m_l^{1/\delta-n+1} f_n(z) g_n(\lambda/m_l)$

Scaling arises from the deep infrared behavior of $P_n(\lambda)$

QCD specific dependence on the ratio of the eigenvalues and the quark mass

Calculating the spectral density of Dirac eigenvalues

- Commonly used method: Lánczos algorithm to calculate the individual low-lying eigenvalues

- We use Chebyshev filtering technique with stochastic estimate of the mode number:

  Mode number for Hermitian operator A in interval $[s,t]$:

$$n_{[s,t]} = \frac{1}{N_V}\sum_{k=1}^{N_V} \xi_k^T h(A)\xi_k,\ N_V,\ \#\text{random vectors},\ h(A) = 1 \text{ inside } [s,t],\ 0 \text{ outside}$$

g_j^p, γ_j are known given $[s,t]$

- Approximate $h(A)$ by Chebyshev polynomials, T_j: $h(A) = \sum_{j=0}^{p} g_j^p \gamma_j T_j(A)$

  For operator restricted to the interval $[-1,1]$, $T_0(A) = I$, $T_1(A) = A$,

  $T_j(A) = 2AT_{j-1}(A) - T_{j-2}(A),\ j \geq 2 \Rightarrow$ use $A = \dfrac{D^\dagger D - (\lambda_{max}^{D^\dagger D} + \lambda_{min}^{D^\dagger D})I/2}{(\lambda_{max}^{D^\dagger D} + \lambda_{min}^{D^\dagger D})I/2}$

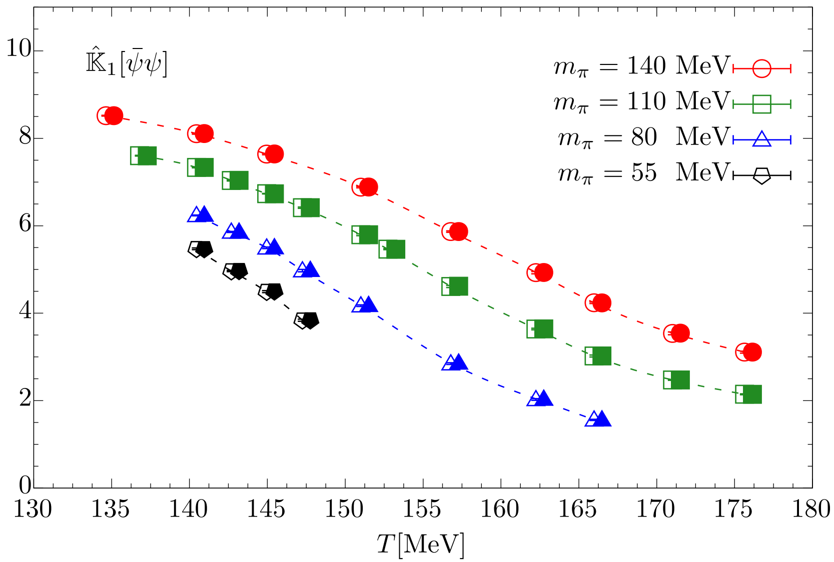
$$\rho_{\mathcal{U}}(\lambda) = \frac{1}{4}\frac{n_{[\lambda-\delta\lambda/2,\lambda+\delta\lambda/2]}}{2\delta\lambda}$$

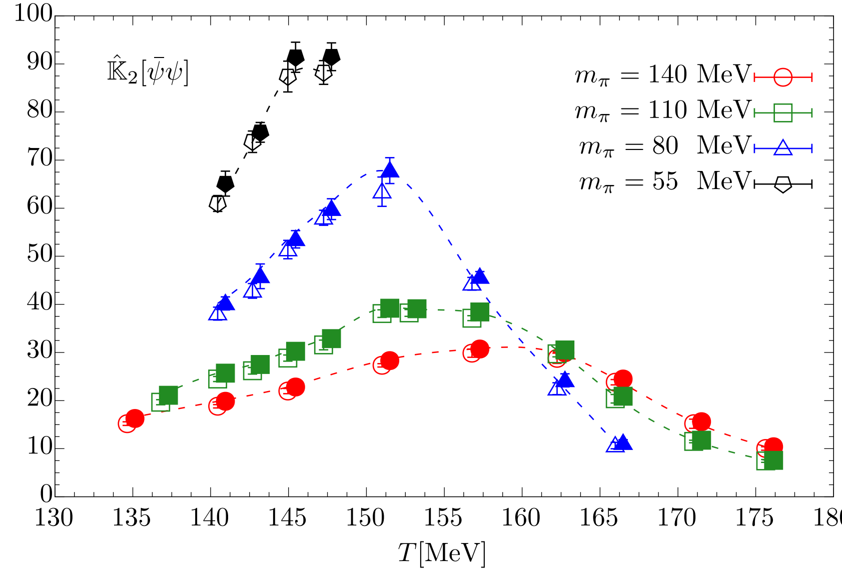
Ding et al, PRL126 (2021) 082001
Zhang et al, Lattice'19, arXiv:2001.05217
Cossu et al, Lattice' 15, arXiv:1601.00744

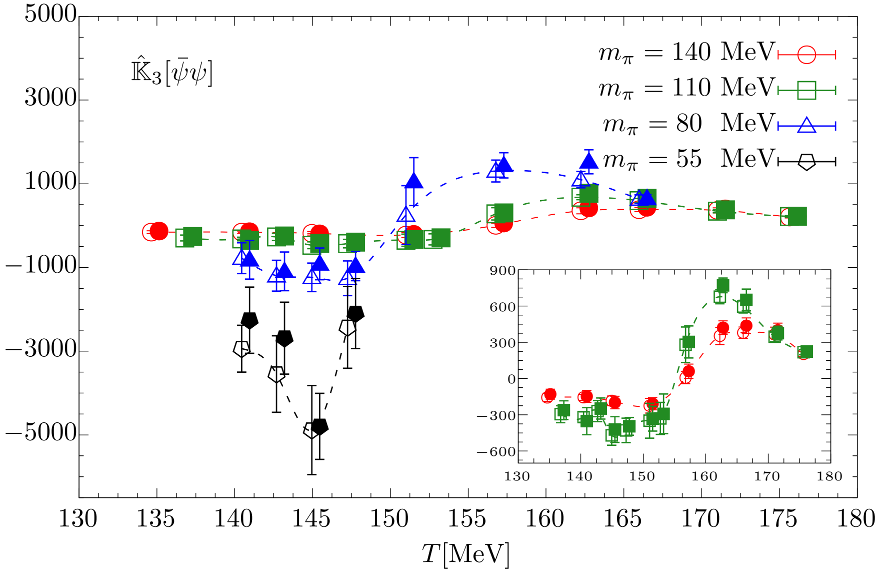
g_j^p suppresses oscillations at the boundary

Di Napoli, Polizzi, Saad, arXiv:1308.4275

Lattice calculation of the cumulants







HISQ action, $N_\sigma^3 \times N_\tau$ lattices
$N_\tau = 8,\ N_\sigma = 32 - 56$
$m_l/m_s = 1/27,\ 1/40,\ 1/80,\ 1/160$
$m_\pi = 140,\ 110,\ 80,\ 55$ MeV
$T = 135 - 176$ MeV

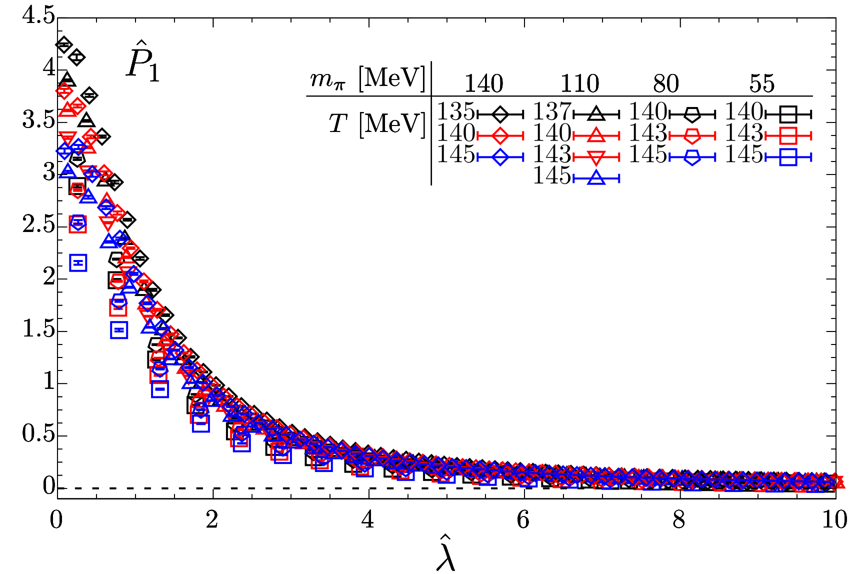
conf: *1.4K-6.8K*

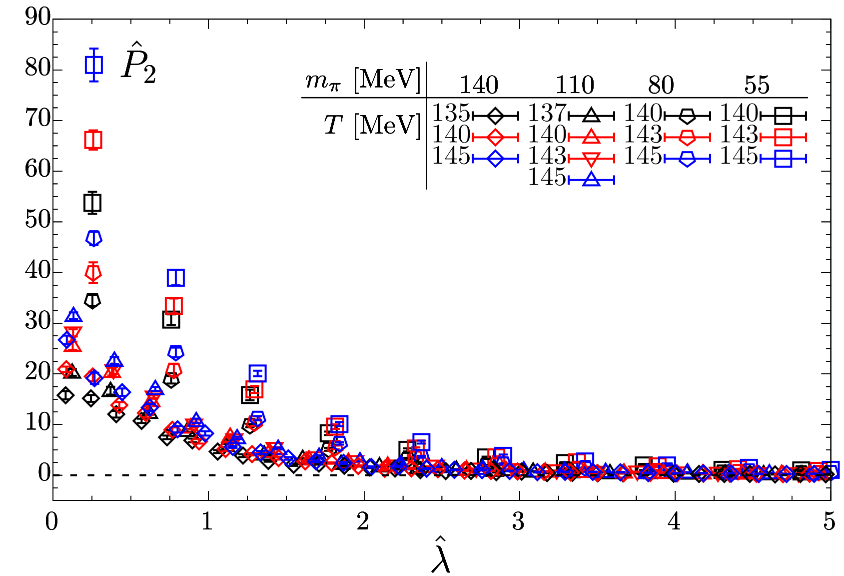
RV: *24* (high *T*)- *96* (low *T*)

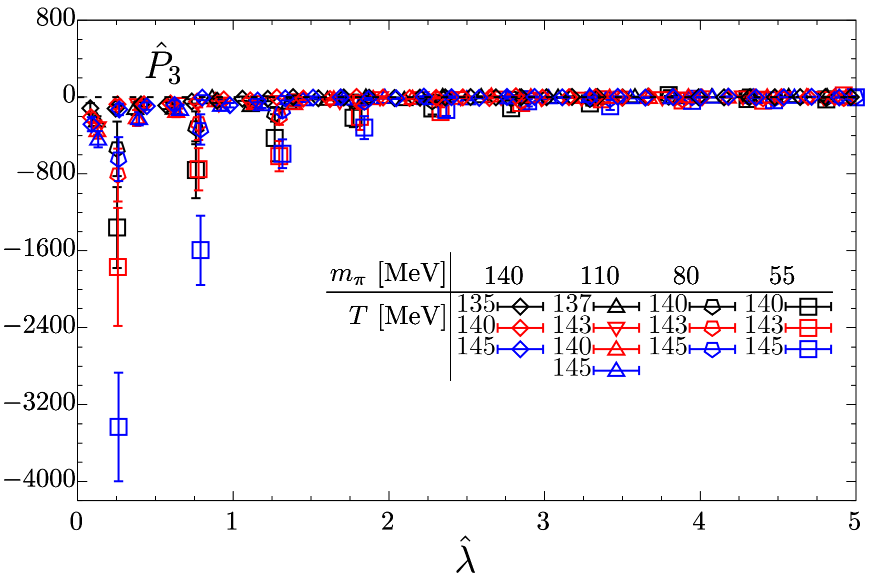
Order of Chebyshev polynomials: *200K*

Open symbols: dir. meas.

Chiral observables and spectrum of Dirac eigenvalues







HISQ action, $N_\sigma^3 \times N_\tau$ lattices

$N_\tau = 8,\ N_\sigma = 32 - 56$

$m_l/m_s = 1/27,\ 1/40,\ 1/80,\ 1/160$

$m_\pi = 140,\ 110,\ 80,\ 55$ MeV

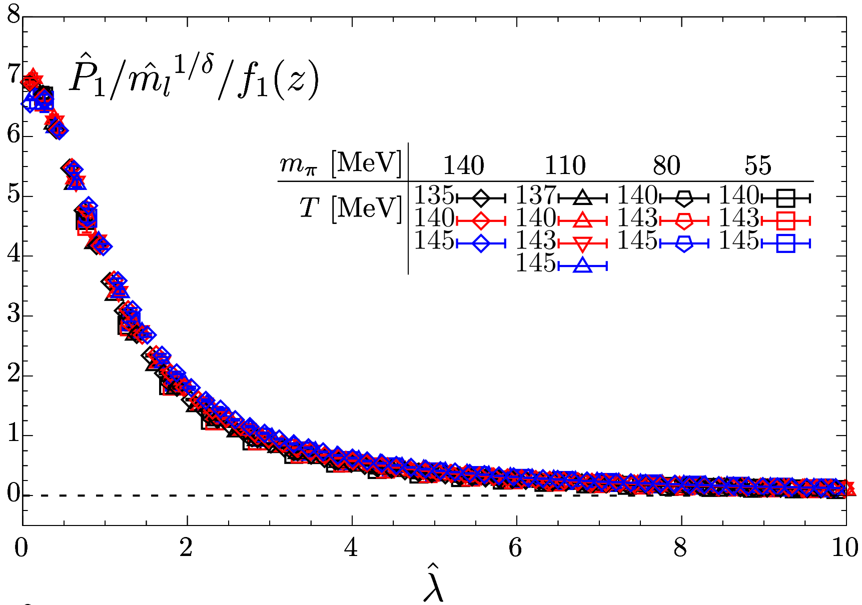
$T = 135 - 176$ MeV

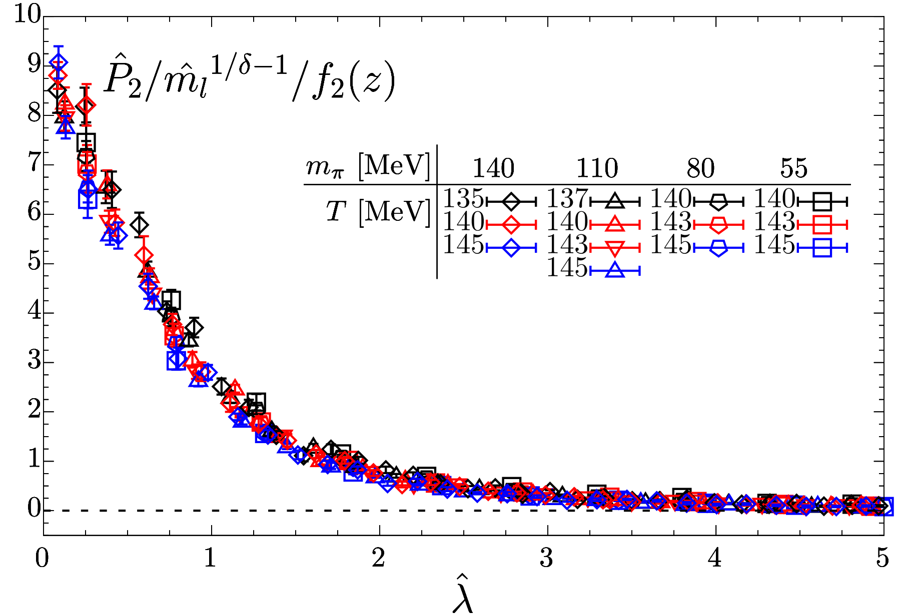
$$\hat{P}_n(\hat{\lambda}) = m_s^2 (m_l/m_s) P_n(\lambda)/T_c^4$$

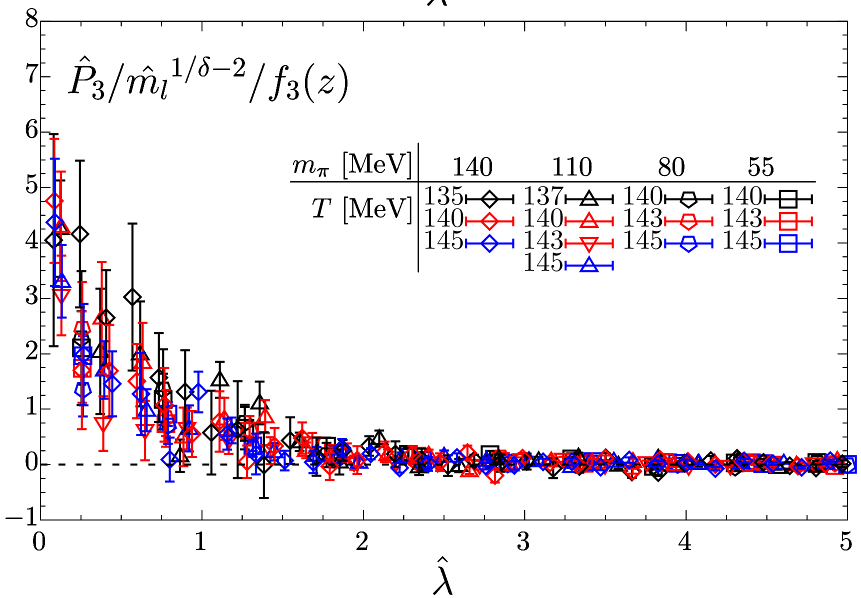
$$\hat{\lambda} = \lambda/m_l$$

Strong quark mass and temperature dependence

Chiral observables and spectrum of Dirac eigenvalues



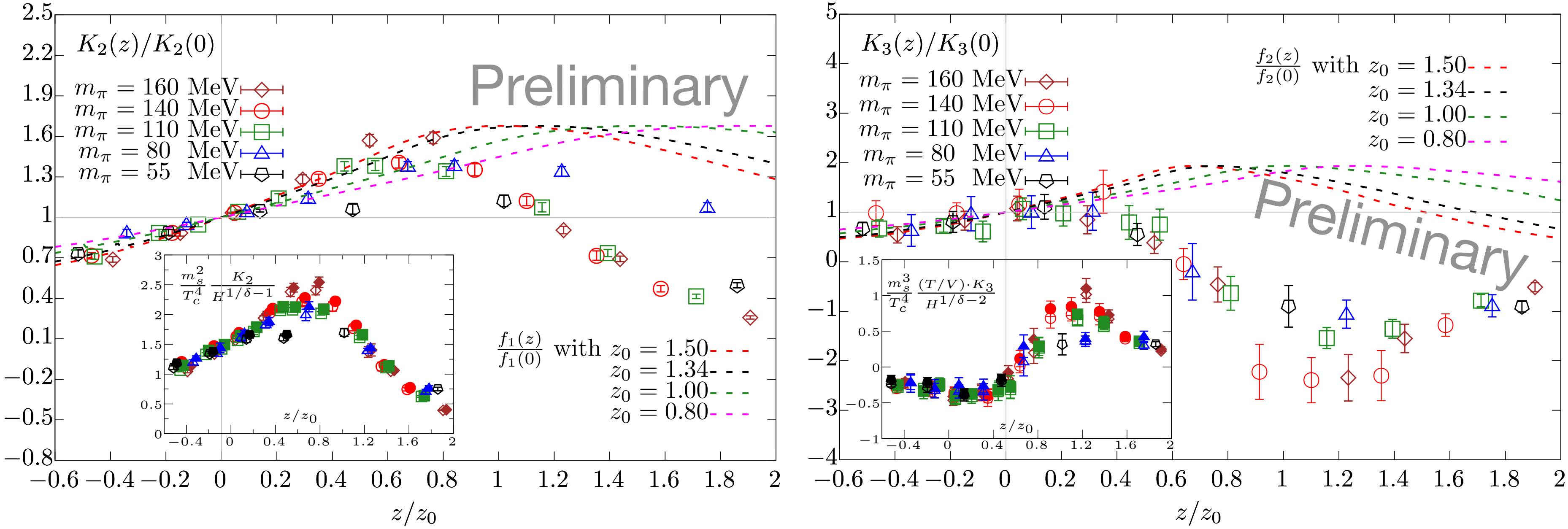




Scaling works for $T \leq 145$ MeV $\sim T_c^0(N_\tau = 8)$

Dirac eigenvalues (energy levels of quarks) know about universality class of the QCD chiral transition

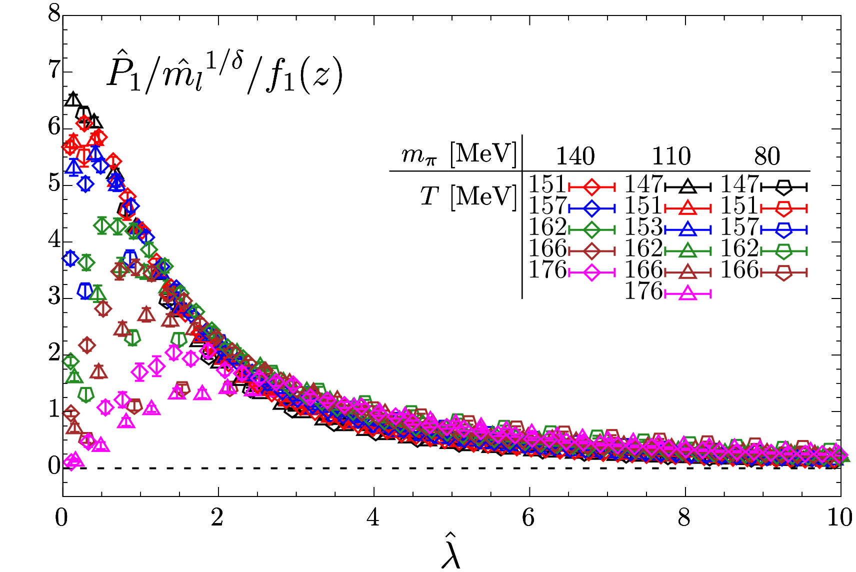
Breakdown of scaling at higher temperatures

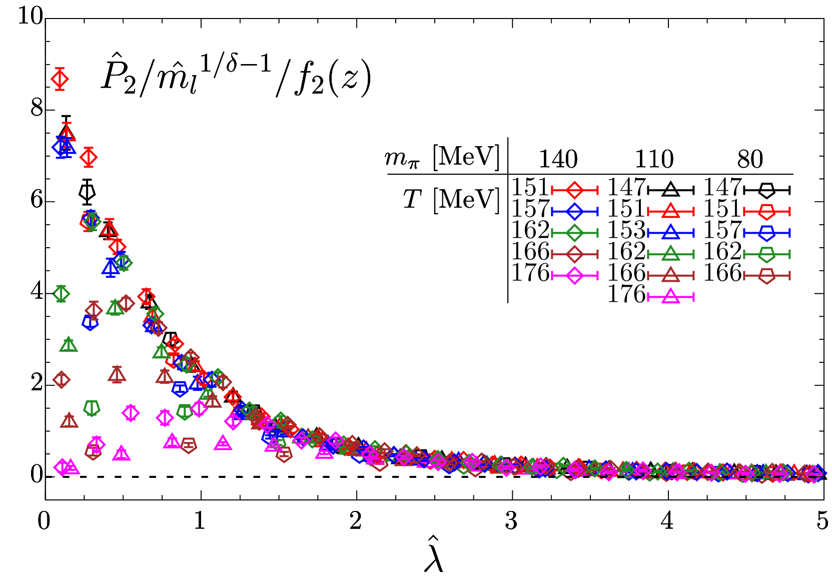


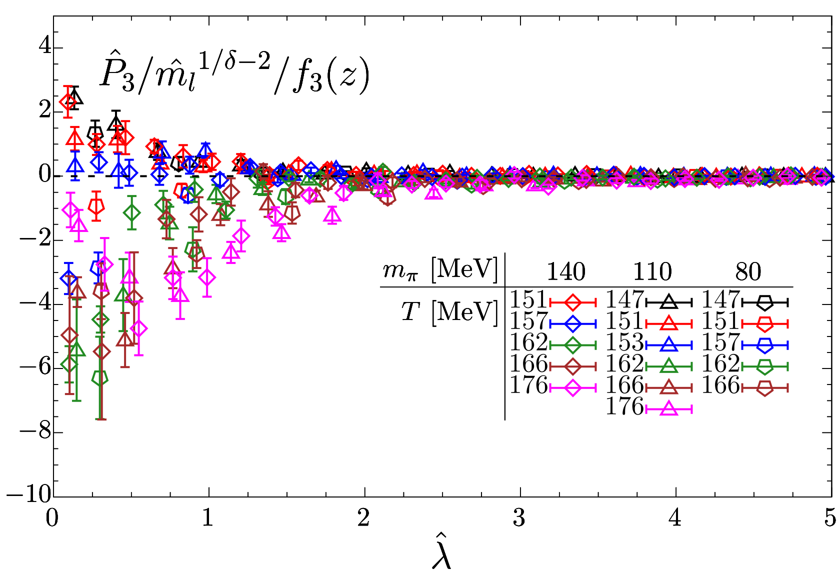
$O(2)$ scaling with $\beta = 0.349$, $\delta = 4.78$, $z_0 = 1.83(9)$, $T_c(N_\tau = 8) = 144.2(6)$ MeV

Clarke et al, PRD 103 (2021) L11501

Breakdown of scaling at higher temperatures







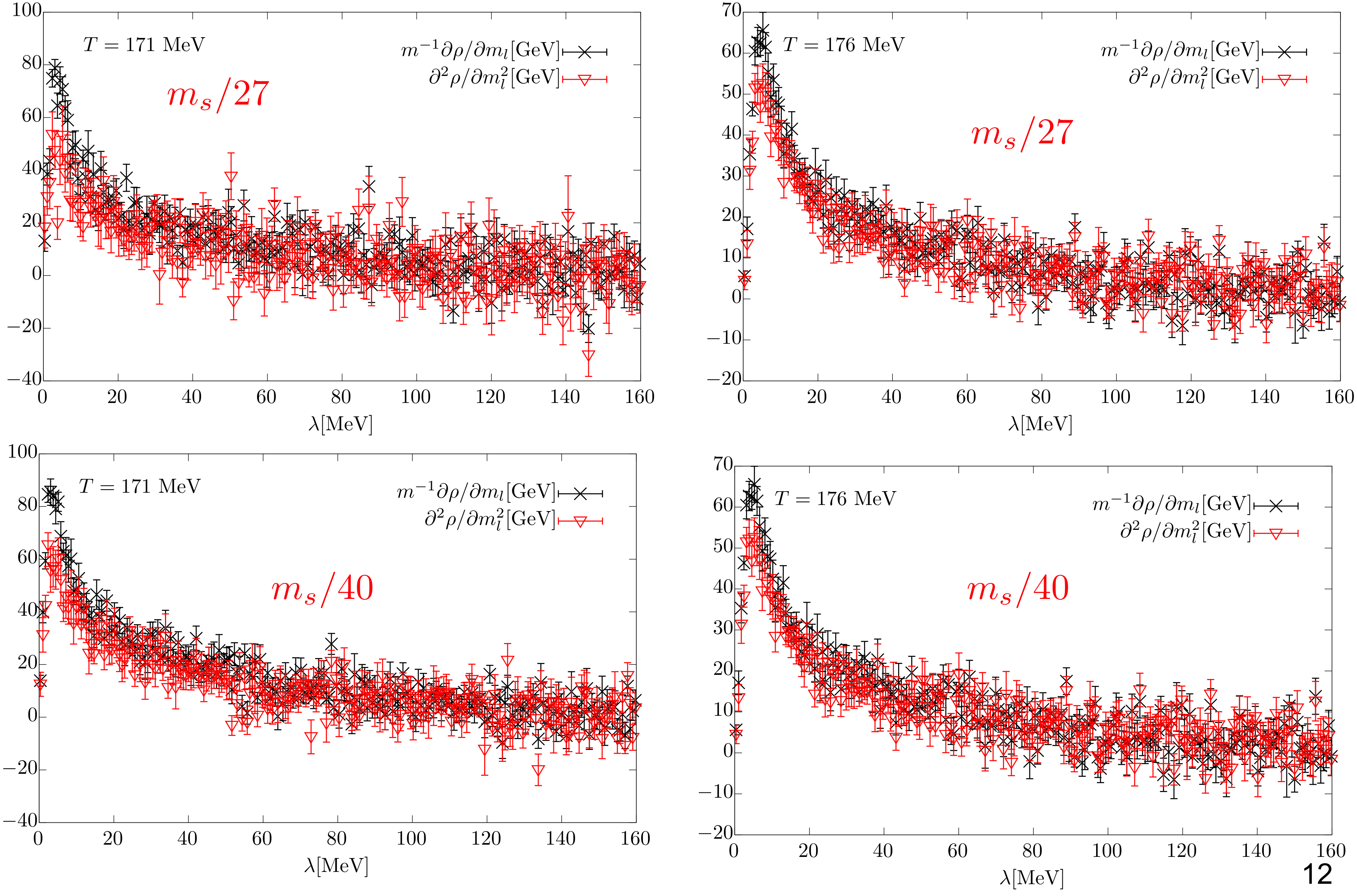
Scaling does not work for *T>147* MeV

Why ?

Transition to instanton gas ?

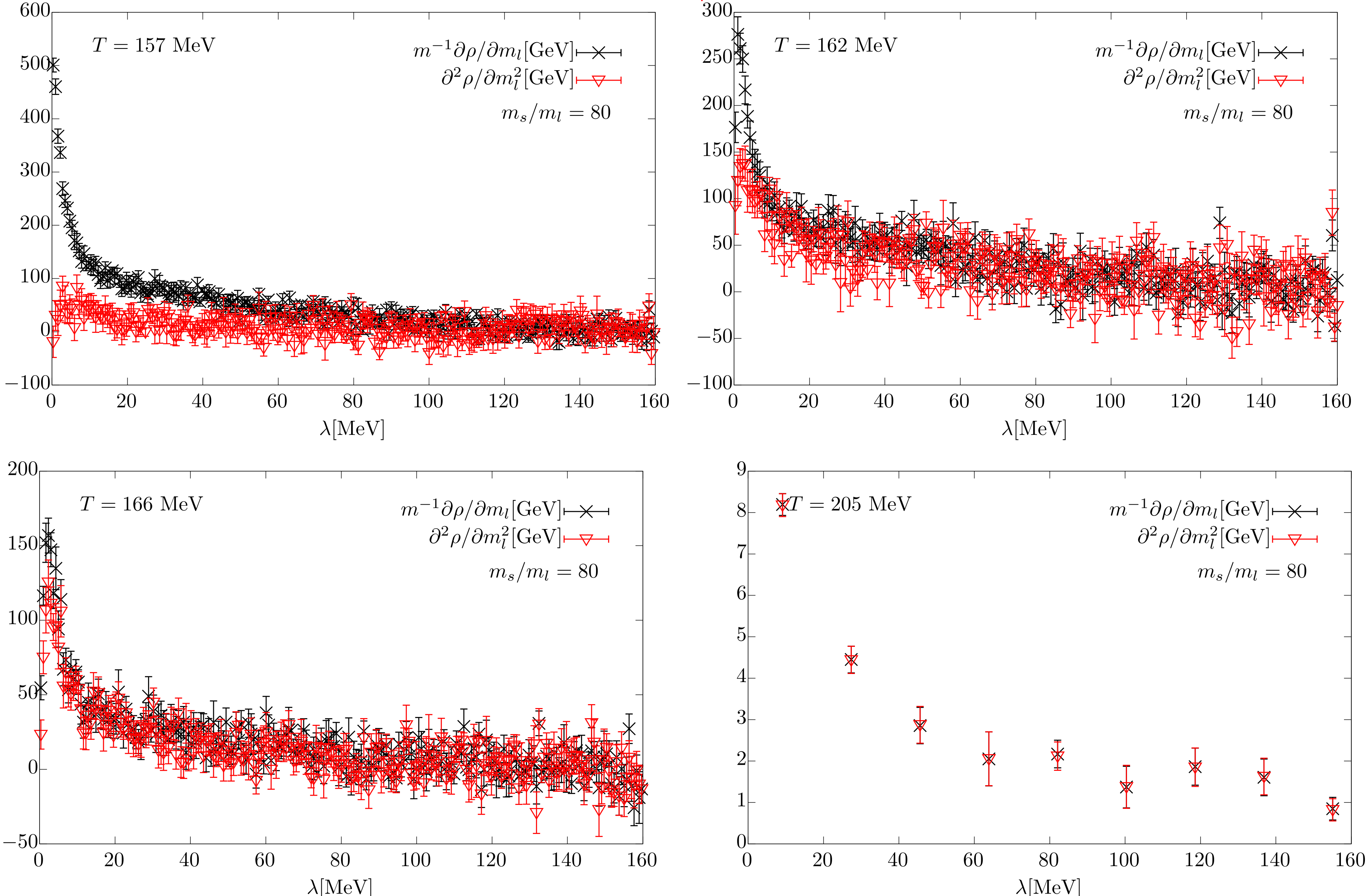
Spectral density of Dirac eigenvalues and instanton gas

Instanton gas: $\rho \sim m_l^2$ vs. $O(N)$ scaling $\rho \sim m_l^n,\ n < 2$



Spectral density of Dirac eigenvalues and instanton gas

$m_s/80$



Summary

- We establish a novel relation:

$$\mathbb{K}_n(\bar{\psi}\psi) = \int [P_{\mathcal{U}}(\lambda_1; m_l) P_{\mathcal{U}}(\lambda_2; m_l) \ldots P_{\mathcal{U}}(\lambda_n; m_l)] \prod_{j=1}^{n} d\lambda_j \equiv \int P_n(\lambda) d\lambda$$

n^{th} cumulant of the chiral condensate

n-point correlations of quark energy spectra

- Obtained a generalization of the Banks-Casher relation:

$$\lim_{m_l \to 0} \mathbb{K}_n(\bar{\psi}\psi) = \lim_{m_l \to 0} = (2\pi)^n \mathbb{K}_n[\rho_{\mathcal{U}}(0)]$$

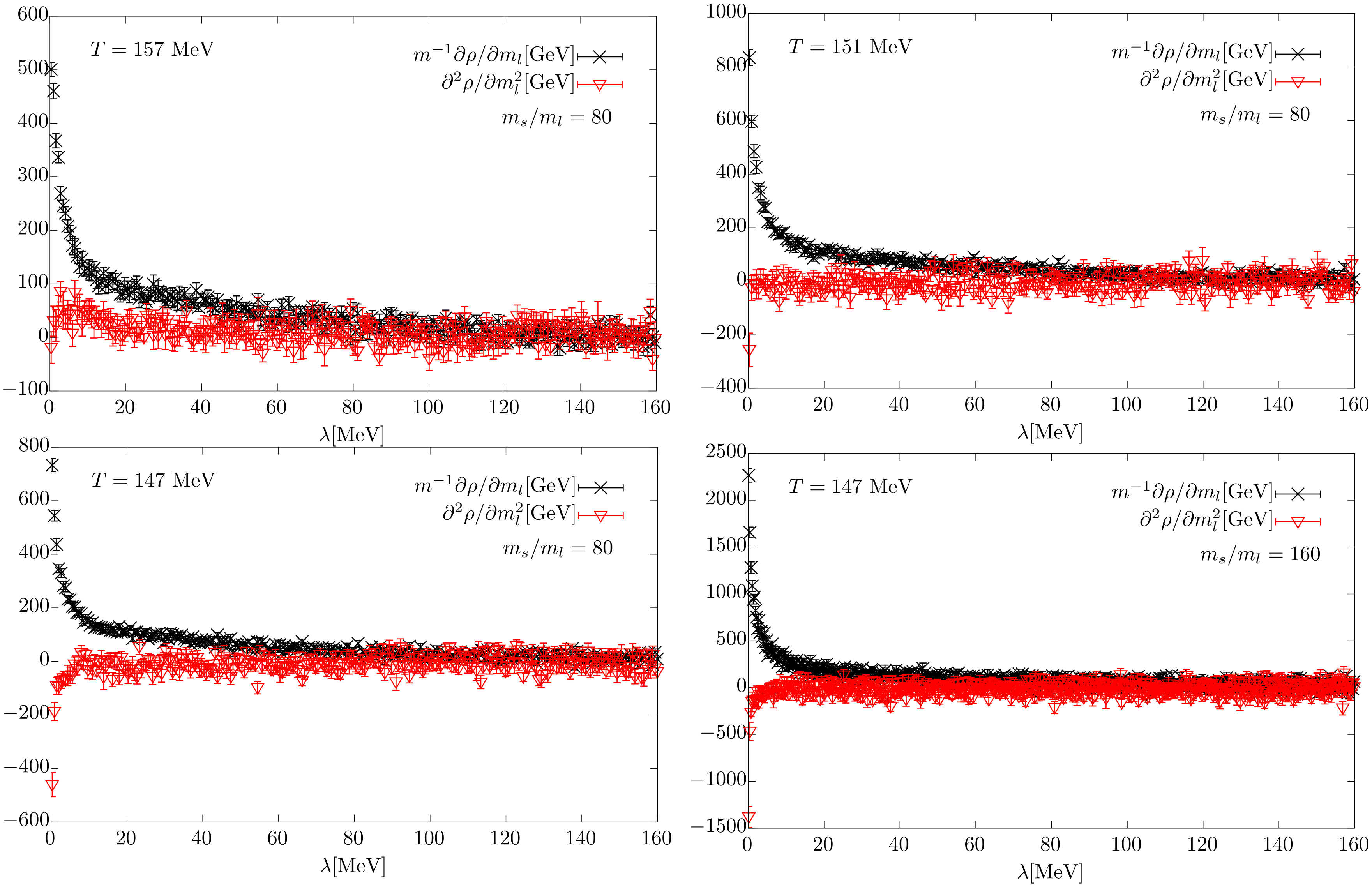
- Found a microscopic encoding of macroscopic universality:

$$P_n(\lambda) = m_l^{1/\delta - n + 1} f_n(z) g_n(\lambda/m_l)$$

- Universal behavior of microscopic QCD Dirac eigenvalues (quark energy levels) in the limit of vanishing quark masses also holds for the physical value of the light quark masses $\Rightarrow$ an alternative method to study universal $O(N)$ scaling

- For $T > 160$ MeV and sufficiently small quark masses we see the onset of instanton gas like behavior, $\rho(\lambda) \sim m_l^2$

Back-up :



$T = 157$ MeV
$m^{-1}\partial\rho/\partial m_l$[GeV]
$\partial^2\rho/\partial m_l^2$[GeV]
$m_s/m_l = 80$
$T = 151$ MeV
$m^{-1}\partial\rho/\partial m_l$[GeV]
$\partial^2\rho/\partial m_l^2$[GeV]
$m_s/m_l = 80$
$T = 147$ MeV
$m^{-1}\partial\rho/\partial m_l$[GeV]
$\partial^2\rho/\partial m_l^2$[GeV]
$m_s/m_l = 80$
$T = 147$ MeV
$m^{-1}\partial\rho/\partial m_l$[GeV]
$\partial^2\rho/\partial m_l^2$[GeV]
$m_s/m_l = 160$
λ[MeV]