Non-perturbative constraints on perturbation theory at finite temperature

(PL, O. Philipsen, 2405.02009)

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CRC–TR 211
Strong-interaction matter under extreme conditions

1. Perturbation theory subtleties for $T > 0$

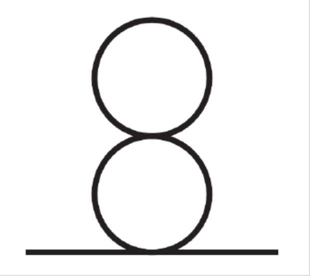
- It has long been understood that the inclusion of temperature $T > 0$ introduces various complications:

  → QCD perturbation theory suffers inconsistencies above a fixed loop order (*"Linde problem"* [*PLB* 96, 1980])

  → Perturbative series have poor convergence properties: need to reorganise the expansion (screened perturbation theory, infinite resummations, functional techniques, ...)

- All of these problems stem from the fact that $T > 0$ perturbation theory introduces a class of diagrams that are not present in the vacuum theory, and these are highly sensitive to the infrared dynamics

e.g. 2-loop order Φ^4 theory:



Includes sub-component

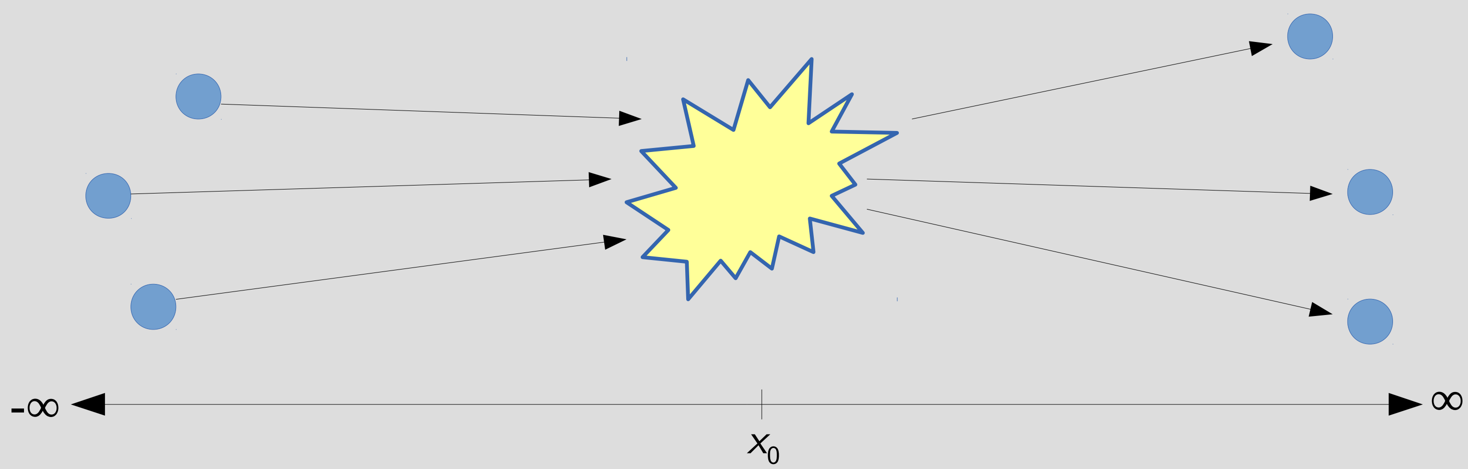
$$T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{1}{(\omega_n^2 + |\vec{p}|^2 + m^2)^2}$$

Diverges for $m \to 0$, even after renormalisation

1. Perturbation theory subtleties for $T > 0$

- These features are often viewed as a purely perturbative feature of finite-temperature QFT, but they may in fact be a symptom of a more fundamental (non-perturbative) constraint:

"*Narnhofer-Requardt-Thirring Theorem*" [Commun. *Math. Phys.* 92, 247 (1983)]



<u>Idea</u>: construct thermal "quasi-particle" in/out scattering states with real dispersion relations $\omega = E(\boldsymbol{p})$ → Conclusion: *the S-matrix must be trivial!*

- So quasi-particles can exist, but only when there are no interactions...

1. Perturbation theory subtleties for $T > 0$

$\rightarrow$ What's the reason for this?

- QFT: *thermal states satisfy the KMS condition, and this gives rise to very different spectral constraints than in the vacuum case*

- Physics: *Dissipative effects of the thermal medium are everywhere-present, so need to take these into account in the definition of scattering states!*

- This has significant implications for perturbation theory: *neither free field, nor quasi-particle propagators with real poles $\omega = E(\boldsymbol{p})$, can form the basis of finite-temperature perturbative expansions* [Landsman, *Ann. Phys.* 186, 141 (1988)]

- Note: these constraints are not restricted to infrared regimes, but also affect systems at low temperatures, or with non-vanishing mass scales (i.e. $T/m \lesssim 1$)

1. Perturbation theory subtleties for $T > 0$

$\rightarrow$ Is there evidence for these constraints?

Yes! Weldon [*PRD* 65 (2002)] showed that the standard perturbative procedure for scalar theories has inconsistencies

- Can classify the singularities occurring in the n-loop perturbative propagator given the singularities of the basic propagator
- If the basic propagators have real poles $\omega = E(\boldsymbol{p})$, at some loop order the perturbative propagator will develop a branch point singularity at these poles, e.g. in Φ^4 theory this occurs at 2-loop order
- This prohibits perturbative corrections to the propagator pole from being computed, hence the standard approach breaks down!

This is a generic feature of *any* perturbative computation that uses propagators with real poles

$$G_R(p) = -\frac{1}{(\omega + i\epsilon)^2 - E(\vec{p})^2}$$

2. Investigation of Φ^4 theory on the lattice

- In principle, the implications of these constraints can also be looked for by comparing perturbative predictions with lattice calculations

  Idea [PL, O. Philipsen, 2405.02009]: perform lattice simulations of basic field correlators in Φ^4 theory and compare these with the predictions of lattice perturbation theory (LPT) $\rightarrow$ see: Montvay & Münster

- In LPT one uses the discretised propagator

$$G(p;a,L,L_\tau) = \frac{1}{\sum_\mu \frac{4}{a^2}\sin^2\left(\frac{ap_\mu}{2}\right) + m_0^2}$$

- Theory is defined on a finite $N_s^3 \times N_\tau$ lattice $\rightarrow$ internal loop integrals are replaced with sums over first Brillouin zone $\mathcal{B}_a = \{p \in \mathbb{R}^4 : -\frac{\pi}{a} < p_\mu \leq \frac{\pi}{a}\}$

- In Φ^4 theory the lattice action is:

$$S = a^4 \sum_{x\in\Lambda}\left[\frac{1}{2}\Delta_\mu^f\phi_0(x)\Delta_\mu^f\phi_0(x) + \frac{m_0^2}{2}\phi_0(x)^2 + \frac{g_0}{4!}\phi_0(x)^4\right]$$

- Here we focus on perturbative calculations of the *spatial correlator*, which is defined:

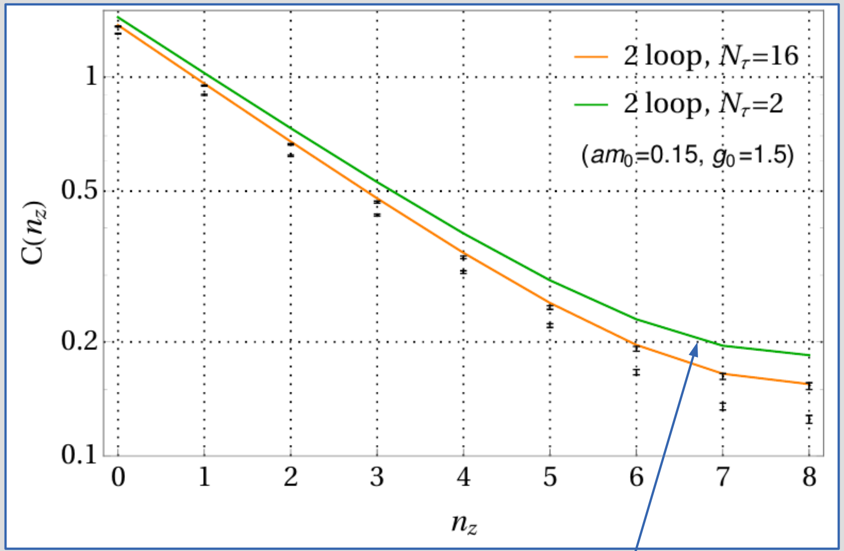
$$C(z;a,N_s,N_\tau) = a^3\sum_{\tau,x,y}\langle\phi(\tau,\vec{x})\phi(0)\rangle.$$

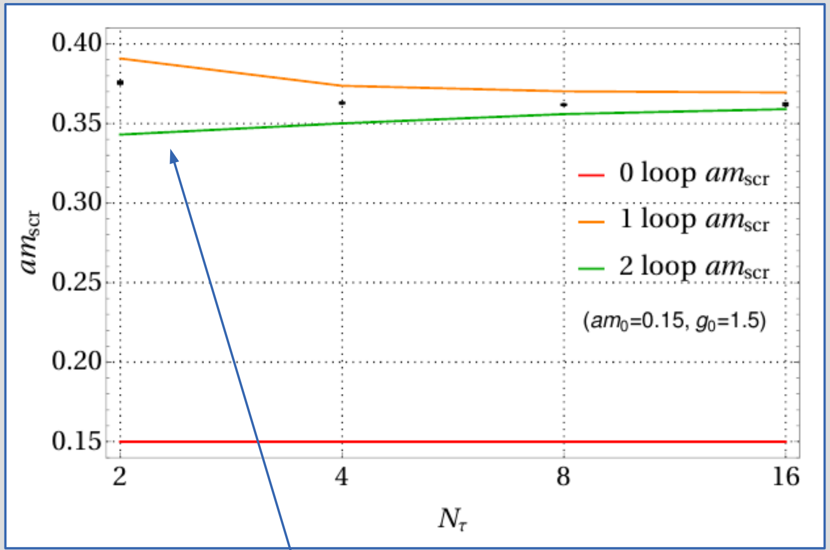
2. Investigation of Φ^4 theory on the lattice

- How do the 2-loop LPT predictions compare with the lattice data as one varies the (lattice) temperature $T = 1/(aN_\tau)$?

  1. For sufficiently small g_0 the perturbative predictions are consistent with the data for all temperatures

  2. For non-negligible g_0 the perturbative predictions for $C(z)$ increasingly deteriorate as T/m increases





For $T/m \sim 1$ the 2-loop predictions deviate drastically from the data

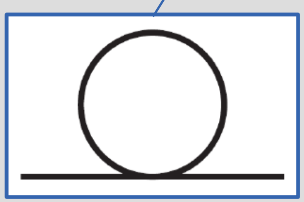
These deviations are also reflected by the screening mass m, defined: $C(z) \sim e^{-mz}$, having the *opposite* T dependence

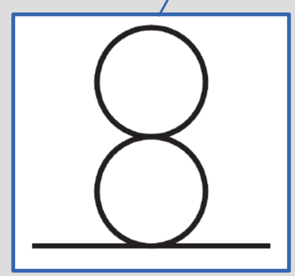
2. Investigation of Φ^4 theory on the lattice

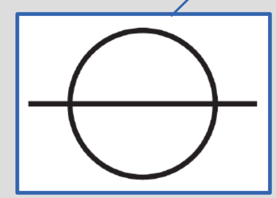
- To establish the origin of these deviations one needs to understand how the 2-loop perturbative predictions are computed

$$C(z;a,N_s,N_\tau)=\frac{1}{N_s}\sum_{k_z=0}^{N_s-1}e^{\frac{2\pi i k_z}{aN_s}z}\frac{a}{4\sin^2\left(\frac{\pi k_z}{N_s}\right)+(am_0)^2+a^2\Pi(\omega_E=p_x=p_y=0,p_z=\frac{2\pi k_z}{aN_s};a,N_s,N_\tau)}.$$

$$a^2\Pi_{(2)}(p;a,N_s,N_\tau)=\frac{g_0}{2}J_1(am_0;N_s,N_\tau)-\frac{g_0^2}{4}J_1(am_0;N_s,N_\tau)J_2(am_0;N_s,N_\tau)-\frac{g_0^2}{6}I_3(p,am_0;N_s,N_\tau)$$







The functions J_n and I_3 are defined:

$$J_n(am_0;N_s,N_\tau)=\frac{1}{N_s^3N_\tau}\sum_{p\in\mathcal{B}_a}\frac{1}{\left[\sum_\mu 4\sin^2\left(\frac{ap_\mu}{2}\right)+(am_0)^2\right]^n},\qquad n=1,2$$

$$I_3(p,am_0;N_s,N_\tau)=\frac{1}{(N_s^3N_\tau)^2}\sum_{q\in\mathcal{B}_a}\sum_{r\in\mathcal{B}_a}\frac{1}{\left[\sum_\mu 4\sin^2\left(p_\mu-\frac{aq_\mu}{2}-\frac{ar_\mu}{2}\right)+(am_0)^2\right]}$$
$$\times\frac{1}{\left[\sum_\nu 4\sin^2\left(\frac{aq_\nu}{2}\right)+(am_0)^2\right]}\frac{1}{\left[\sum_{\Lambda_a} 4\sin^2\left(\frac{ar_\lambda}{2}\right)+(am_0)^2\right]}$$

2. Investigation of Φ^4 theory on the lattice

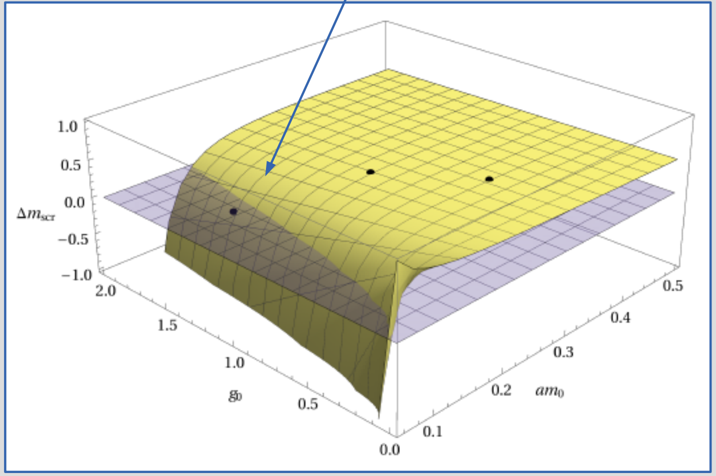
- Numerically, one finds that the sunset diagram contribution is sub-dominant, and that the behaviour of $C(z)$ is largely controlled by the interplay between the tadpole and cactus diagrams

- As one varies (am_0, g_0) there are certain choices for which the self-energy at any fixed N_τ is ***smaller*** than when $N_\tau \to \infty$ (i.e. $T = 0$)

  $\rightarrow$ This implies that the screening mass m_{scr}, which is defined via the (continuum) large-z behaviour: $C(z) \sim \exp(-m_{scr}\, z)$, ***decreases*** with T

$$\Delta m_{\rm scr} = \frac{\frac{g_0}{2} J_1(am_0; N_s, N_\tau = 1) - \frac{g_0^2}{4} J_1(am_0; N_s, N_\tau = 1) J_2(am_0; N_s, N_\tau = 1)}{\frac{g_0}{2} J_1(am_0; N_s, N_\tau = \infty) - \frac{g_0^2}{4} J_1(am_0; N_s, N_\tau = \infty) J_2(am_0; N_s, N_\tau = \infty)} - 1,$$

Predictions with (am_0, g_0) closer to the region $\Delta m_{scr} < 0$ deviate further from the data



- When $\Delta m_{scr} < 0$, and the sunset is sufficiently small, then: $m_{scr}\,(T = 0) > m_{scr}\,(T > 0)$

- This behaviour is ***opposite*** to physical expectations (and the lattice data!)

2. Investigation of Φ^4 theory on the lattice

- The proximity to the region $\Delta m_{\text{scr}} < 0$ depends on the structure of the propagators entering the loop calculations
- From the plot one can see that for fixed g_0 one will *always* move into this region when am_0 is sufficiently small

→ This can be understood by the fact that for small am_0 the zero-mode contributions dominate the sums in J_n, and these have the form:

$$J_1(am_0; N_s, N_\tau) \sim \frac{1}{N_s^3 N_\tau}\frac{1}{(am_0)^2}, \quad J_2(am_0; N_s, N_\tau) \sim \frac{1}{N_s^3 N_\tau}\frac{1}{(am_0)^4}.$$

→ The cactus diagram outcompetes the tadpole for small enough am_0

Conclusion: the deviations in the two-loop perturbative predictions are driven by the analytic structure of the free field propagators → this is consistent with the constraints of the NRT theorem and the findings of Weldon [*PRD* 65 (2002)]

3. A potential resolution

- The NRT constraints and previous results suggest that one needs an alternative approach to scalar perturbation theory when $T > 0$

  Idea: Start with propagators that consistently take thermal effects into account from the outset, i.e. non-real singularities

  $\rightarrow$ ***But what form should these propagators take?***

- There's theoretical evidence to suggest that the thermal medium contains discrete particle-like excitations [Bros, Buchholz, *NPB* 627 (2002)]

$$\rho(\omega,\vec{p}) = \int_0^\infty ds \int \frac{d^3\vec{u}}{(2\pi)^2}\, \epsilon(\omega)\, \delta\big(\omega^2 - (\vec{p}-\vec{u})^2 - s\big)\, \widetilde{D}_\beta(\vec{u},s)$$

"Thermoparticle"

$$\rightarrow \quad \widetilde{D}_{m,\beta}(\vec{u})\, \delta(s-m^2)$$

$\rightarrow$ Thermoparticles reduce to those of a vacuum particle with mass m when $T \rightarrow 0$

$\rightarrow$ "Damping factor" $\widetilde{D}_\beta(\boldsymbol{u})$ results in thermally-broadened peaks in the spectral function: this parametrises the effects of collisional broadening

3. A potential resolution

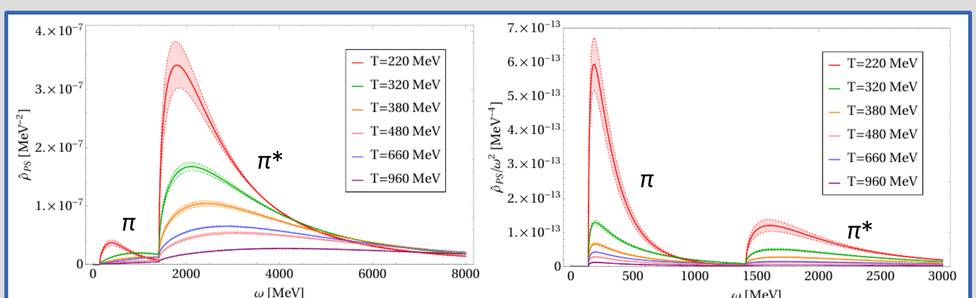
- Thermoparticle components have distinctive properties:

1. They dominate the large-time behaviour of $\langle \phi(x_0, \vec{x}) \phi(0) \rangle$

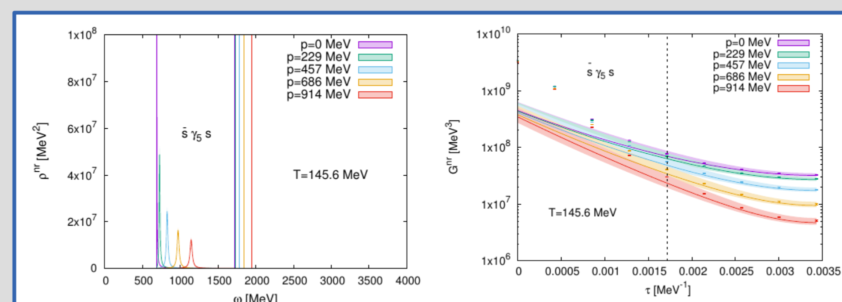
2. Along the direction $\boldsymbol{x} = \boldsymbol{v}\, x_0$ they behave like: $D_{m,\beta}(\vec{v} x_0)\, |x_0|^{-3/2}$ so their damped behaviour depends on their velocity [Bros, Ann. H. Poincare 4, (2003)]

3. They evade the consequences of the NRT theorem since they no longer have a sharp dispersion law

→ There's also (mounting) evidence from lattice QCD data that these excitations are present in various (meson) spectral functions



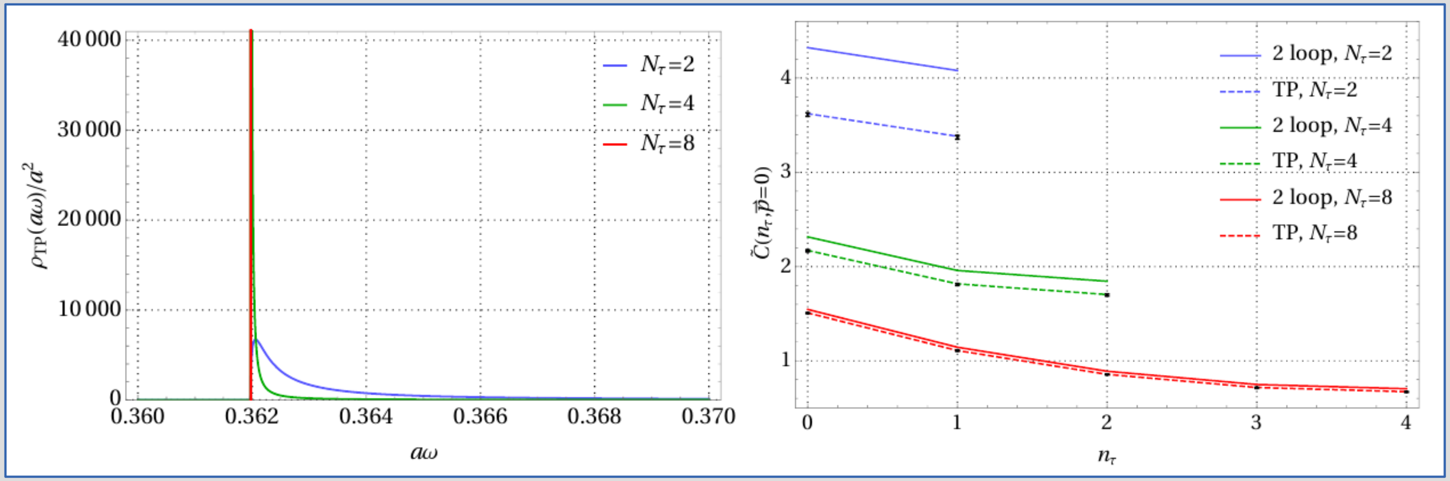
Light-light pseudo-scalar meson (pion) channel [P.L., O. Philipsen, *JHEP* 10, 161 (2022)]



Light-strange (kaon) and strange-strange (eta) pseudo-scalar meson channels [D. Bala, O. Kaczmarek, P. L., O. Philipsen, and T. Ueding, *JHEP* 05, 332 (2024)]

3. A potential resolution

- Thermoparticles dominate the large-time behaviour of correlators and are therefore natural candidates for describing scattering states when $T > 0$
- Scattering states determine the form of the basic field propagators used in any perturbative QFT expansion $\rightarrow$ *Thermoparticle propagators could form the basis of a consistent $T > 0$ perturbative expansion*
- Note: It has been shown that these propagators can be *uniquely* fixed by the dynamical equations of the theory [Bros, Buchholz, 2002]
- For Φ^4 theory there appear to be signatures of these excitations in the data



See: [PL, O. Philipsen, 2405.02009]

Summary & outlook

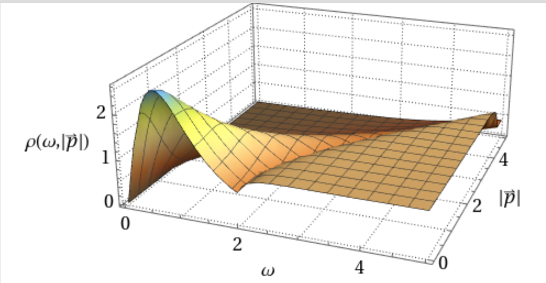
- Perturbation theory has several known issues related to the infrared structure of certain classes of diagrams when $T > 0$

- This seems to be a realisation of a more fundamental constraint: non-trivial scattering at $T > 0$ is incompatible with real dispersion laws (*NRT theorem*)

- There is now both analytic (Weldon, 2002) and numerical (PL, Philipsen, 2024) evidence for this, even for massive theories, and at small T

- A potential resolution of these issues is to use propagators which don't have real dispersion laws $\rightarrow$ thermoparticles are a natural candidate

- The analytic structure of thermoparticle propagators can be derived for specific QFTs (Bros, Buchholz, 2002), and therefore might provide a consistent perturbative framework when $T > 0$. *Stay tuned!*



Backup: *(Axiomatic) QFT in a nutshell*

- QFT framework defined by a core set of axioms:

Axiom 1 (Hilbert space structure). *The states of the theory are rays in a Hilbert space $\mathcal{H}$ which possesses a continuous unitary representation $U(a,\alpha)$ of the Poincaré spinor group $\overline{\mathscr{P}_+^{\uparrow}}$.*

Axiom 2 (Spectral condition). *The spectrum of the energy-momentum operator P^μ is confined to the closed forward light cone $\overline{V}^+ = \{p^\mu \mid p^2 \geq 0,\, p^0 \geq 0\}$, where $U(a,1) = e^{iP^\mu a_\mu}$.*

Axiom 3 (Uniqueness of the vacuum). *There exists a unit state vector $|0\rangle$ (the vacuum state) which is a unique translationally invariant state in $\mathcal{H}$.*

Axiom 4 (Field operators). *The theory consists of fields $\varphi^{(\kappa)}(x)$ (of type (κ)) which have components $\varphi_l^{(\kappa)}(x)$ that are operator-valued tempered distributions in $\mathcal{H}$, and the vacuum state $|0\rangle$ is a cyclic vector for the fields.*

Axiom 5 (Relativistic covariance). *The fields $\varphi_l^{(\kappa)}(x)$ transform covariantly under the action of $\overline{\mathscr{P}_+^{\uparrow}}$:*

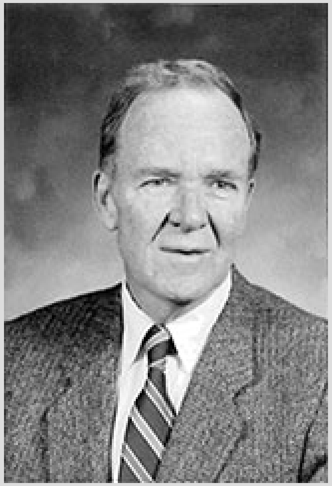
$$U(a,\alpha)\varphi_i^{(\kappa)}(x)U(a,\alpha)^{-1} = S_{ij}^{(\kappa)}(\alpha^{-1})\varphi_j^{(\kappa)}(\Lambda(\alpha)x + a)$$

where $S(\alpha)$ is a finite dimensional matrix representation of the Lorentz spinor group $\overline{\mathscr{L}_+^{\uparrow}}$, and $\Lambda(\alpha)$ is the Lorentz transformation corresponding to $\alpha \in \overline{\mathscr{L}_+^{\uparrow}}$.

Axiom 6 (Local (anti-)commutativity). *If the support of the test functions f, g of the fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$ are space-like separated, then:*

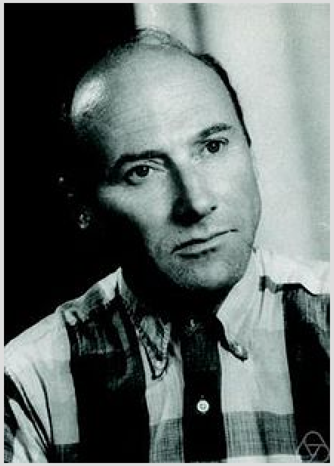
$$[\varphi_l^{(\kappa)}(f), \varphi_m^{(\kappa')}(g)]_\pm = \varphi_l^{(\kappa)}(f)\varphi_m^{(\kappa')}(g) \pm \varphi_m^{(\kappa')}(g)\varphi_l^{(\kappa)}(f) = 0$$

when applied to any state in $\mathcal{H}$, for any fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$.



A. Wightman

[R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and all that* (1964).]



R. Haag

[R. Haag, *Local Quantum Physics*, Springer-Verlag (1996).]

Backup: *Particle signatures in vacuum*

- Correlation functions $\langle 0|\varphi_{l_1}^{(\kappa_1)}(x_1)\cdots\varphi_{l_n}^{(\kappa_n)}(x_n)|0\rangle$ encode ***all*** of the dynamical information, including the properties of particles!
- For the purpose of this talk I will focus on (scalar) 2-point correlation functions, in particular the *spectral function*, defined by

$$\rho(\omega,\vec{p}) = \int d^4x\, e^{i(\omega x_0 - \vec{p}\cdot\vec{x})}\langle[\phi(x),\phi(0)]\rangle$$

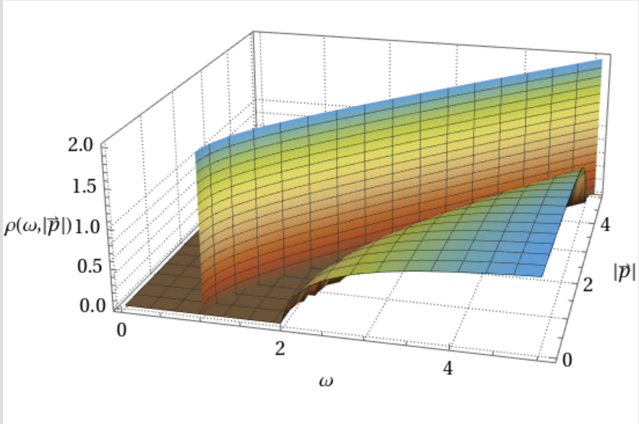
- The simplest case: a non-interacting particle with mass *m*

$$\rho(\omega,\vec{p}) = 2\pi\epsilon(\omega)\delta(\omega^2 - |\vec{p}|^2 - m^2)$$

... *and when we add interactions*

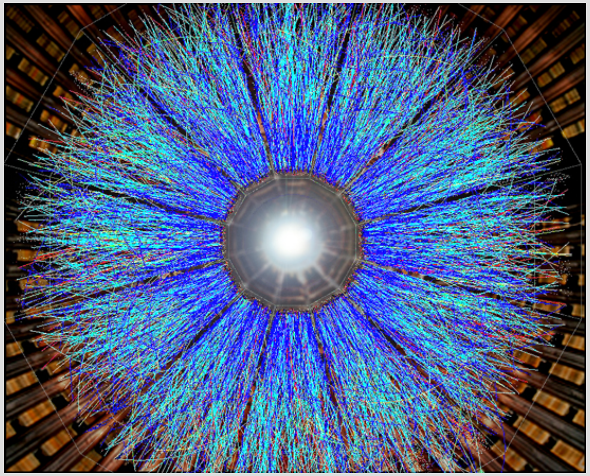
Delta peak → stable particle component

Continuum onset → multi-particle states

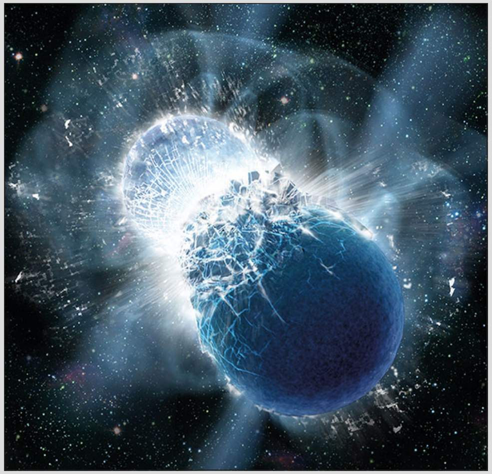


Backup: *Beyond the vacuum*

- To describe physical phenomena in "extreme environments" one must understand of how QFT applies to systems that are hot, dense, or both



[Brookhaven National Lab]



[Skyworks Digital Inc.]

- Therefore need to figure out how the inclusion of temperature $T = 1/\beta$ or density modifies the standard QFT assumptions, and what effect this has on the correlation functions

$\rightarrow$ Let's stick to the case $T > 0$ and vanishing density

Backup: *Beyond the vacuum*

- **Idea**: Look for a generalisation of the standard axioms which is compatible with $T > 0$, and approaches the vacuum case for $T \to 0$

Axiom 1 (**Hilbert space structure**). *The states of the theory are rays in a Hilbert space $\mathcal{H}$ which possesses a continuous unitary representation $U(a, \alpha)$ of the Poincaré spinor group $\overline{\mathscr{P}_+^\uparrow}$.*

→ H_β **is defined for fixed** $\beta = 1/T$

Axiom 2 (**Spectral condition**). *The spectrum of the energy-momentum operator P^μ is confined to the closed forward light cone $\overline{V}^+ = \{p^\mu \mid p^2 \geq 0,\, p^0 \geq 0\}$, where $U(a, 1) = e^{iP^\mu a_\mu}$.*

→ **Replaced by the KMS condition**

$$\langle \Omega_\beta | \phi(x_1) \cdots \phi(x_k) \phi(x_{k+1}) \cdots \phi(x_n) | \Omega_\beta \rangle = \langle \Omega_\beta | \phi(x_{k+1}) \cdots \phi(x_n) \phi(x_1 + i(\beta, \vec{0})) \cdots \phi(x_k + i(\beta, \vec{0})) | \Omega_\beta \rangle$$

Axiom 3 (**Uniqueness of the vacuum**). *There exists a unit state vector $|0\rangle$ (the vacuum state) which is a unique translationally invariant state in $\mathcal{H}$.*

→ **Instead, thermal background state** $|\Omega_\beta>$

Axiom 4 (**Field operators**). *The theory consists of fields $\varphi^{(\kappa)}(x)$ (of type (κ)) which have components $\varphi_l^{(\kappa)}(x)$ that are operator-valued tempered distributions in $\mathcal{H}$, and the vacuum state $|0\rangle$ is a cyclic vector for the fields.*

✓ **Fields are still distributions**

Axiom 5 (**Relativistic covariance**). *The fields $\varphi_l^{(\kappa)}(x)$ transform covariantly under the action of $\overline{\mathscr{P}_+^\uparrow}$:*

$$U(a, \alpha) \varphi_i^{(\kappa)}(x) U(a, \alpha)^{-1} = S_{ij}^{(\kappa)}(\alpha^{-1}) \varphi_j^{(\kappa)}(\Lambda(\alpha)x + a)$$

where $S(\alpha)$ is a finite dimensional matrix representation of the Lorentz spinor group $\overline{\mathscr{L}_+^\uparrow}$, and $\Lambda(\alpha)$ is the Lorentz transformation corresponding to $\alpha \in \overline{\mathscr{L}_+^\uparrow}$.

→ **The fields no longer transform under general unitary Lorentz transformations**

Axiom 6 (**Local (anti-)commutativity**). *If the support of the test functions f, g of the fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$ are space-like separated, then:*

$$[\varphi_l^{(\kappa)}(f), \varphi_m^{(\kappa')}(g)]_\pm = \varphi_l^{(\kappa)}(f)\varphi_m^{(\kappa')}(g) \pm \varphi_m^{(\kappa')}(g)\varphi_l^{(\kappa)}(f) = 0$$

when applied to any state in $\mathcal{H}$, for any fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$.

✓ **Causality is unaffected by the properties of the background state. *This is important!***

Backup: *Beyond the vacuum*

- For $T > 0$ there are some immediate modifications:

  - **Lorentz invariance** ✘ $\rightarrow$ but can retain rotational invariance
  - **Spectral condition** ✘ $\rightarrow$ replaced by equilibrium (KMS) condition
  - **Field locality (causality)** ✓ $\rightarrow$ this is important!

- Taking all of these $T > 0$ constraints into account implies that the spectral function has the representation*

$$\rho(\omega,\vec{p}) = \int_0^\infty ds \int \frac{d^3\vec{u}}{(2\pi)^2}\, \epsilon(\omega)\, \delta\big(\omega^2 - (\vec{p}-\vec{u})^2 - s\big)\, \widetilde{D}_\beta(\vec{u},s)$$

"Thermal spectral density"

This is the $T > 0$ generalisation of the textbook *Källén-Lehmann* representation!

$$\rho(\omega,\vec{p}) = 2\pi\epsilon(\omega)\int_0^\infty ds\, \delta\big(p^2 - s\big)\, \varrho(s)$$

- $T > 0$ effects amount to understanding: $\rho(s) \rightarrow \widetilde{D}_\beta(\boldsymbol{u},s)$, which tell us about the possible excitations that can exist in a thermal medium

* See: *J. Bros and D Buchholz, Z. Phys. C* 55 (1992), *Ann. Inst. H.Poincare Phys.Theor.* 64 (1996)

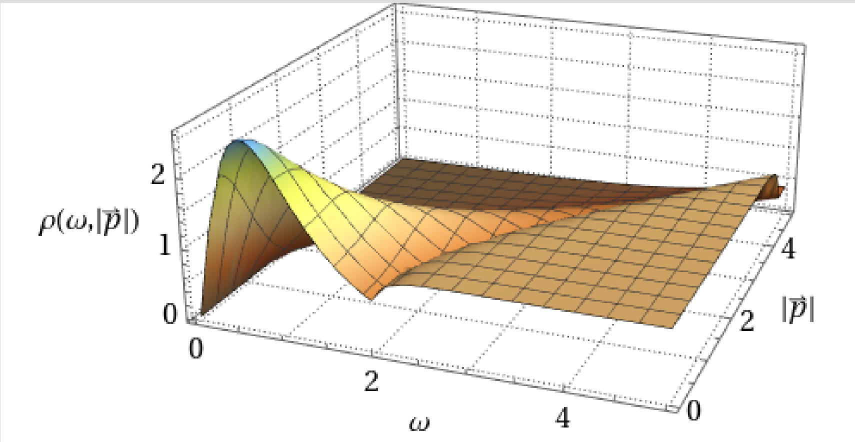
Backup: *A thermal particle?* → *"Thermoparticle"*

Proposition: the medium contains "Thermoparticles": particle-like excitations which differ from collective quasi-particle modes, and show up as ***discrete*** contributions to $\widetilde{D}_\beta(\boldsymbol{u},s)$ [Bros, Buchholz, *NPB* 627 (2002)]

$$\widetilde{D}_\beta(\vec{u},s) = \widetilde{D}_{m,\beta}(\vec{u})\,\delta(s-m^2) + \widetilde{D}_{c,\beta}(\vec{u},s)$$

→ Thermoparticle components $\widetilde{D}_\beta(\boldsymbol{u})\delta(s\text{-}m^2)$ reduce to those of a vacuum particle with mass m in the limit $T \to 0$

→ Non-trivial "Damping factor" $\widetilde{D}_\beta(\boldsymbol{u})$ results in thermally-broadened peaks in the spectral function: this parametrises the effects of collisional broadening



→ Component $\widetilde{D}_{c,\beta}(\boldsymbol{u},s)$ contains all other types of excitations, including those that are *continuous* in s

Backup: *Perturbation theory setup*

- Perturbation theory for $T = 1/\beta > 0$ can be performed using either real or imaginary-time propagators. Both have their advantages and disadvantages (see: Kapusta & Gale, Le Bellac)

- Here we focus on the imaginary time τ formalism. The basic idea is that one uses imaginary time "Matsubara" propagators which have discrete energies $\omega_n = 2\pi n/\beta$

$$G(\omega_n, \vec{p}) = \frac{1}{\omega_n^2 + |\vec{p}|^2 + m^2}$$

- This discretisation comes about because the Euclidean 2-point function $W_E(\tau, \boldsymbol{x})$ is β-periodic in τ (due to the KMS condition)

  $\rightarrow$ At the end one analytically continues the result: $i\omega_n \rightarrow \omega + i\varepsilon$

- The perturbation theory rules are analogous to the $T = 0$ case, except now each internal line is integrated with the measure:

$$T \sum_n \int \frac{d^3p}{(2\pi)^3}$$

Backup: *Perturbation theory results*

- Results for L=16:

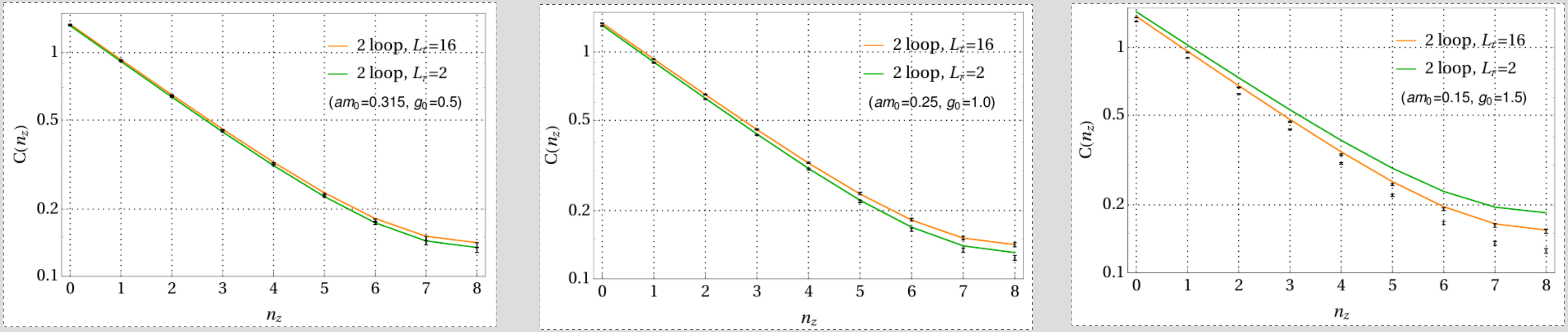
For small coupling the results agree with data for all T/m

For large coupling the vacuum-like data differs from prediction

→ Higher-loop corrections needed!

$L^3 \times L_\tau$	(am_0, g_0)	m/Λ_c	T/m	$(\chi^2/\text{d.o.f.})_{\text{2-loop}}$	$\Delta^\ell_{\text{1-loop}}$ [%]	$\Delta^\ell_{\text{2-loop}}$ [%]
$16^3 \times 16$	$(0.315, 0.5)$	0.118	0.17	1.7	0.4 (0.6)	0.8 (0.6)
$16^3 \times 8$	$(0.315, 0.5)$	0.118	0.34	0.4	0.2 (0.6)	0.2 (0.6)
$16^3 \times 4$	$(0.315, 0.5)$	0.118	0.68	0.3	1.0 (0.6)	0.4 (0.6)
$16^3 \times 2$	$(0.315, 0.5)$	0.118	1.35	0.5	1.2 (0.6)	0.4 (0.6)
$16^3 \times 16$	$(0.25, 1.0)$	0.117	0.17	0.1	2.0 (0.6)	0.2 (0.6)
$16^3 \times 8$	$(0.25, 1.0)$	0.117	0.34	0.7	1.1 (0.6)	0.8 (0.6)
$16^3 \times 4$	$(0.25, 1.0)$	0.117	0.68	0.1	2.4 (0.6)	0.1 (0.6)
$16^3 \times 2$	$(0.25, 1.0)$	0.117	1.36	3.9	3.3 (0.6)	0.9 (0.6)
$16^3 \times 16$	$(0.15, 1.5)$	0.115	0.17	10.3	4.1 (0.6)	2.3 (0.6)
$16^3 \times 8$	$(0.15, 1.5)$	0.115	0.35	49.7	4.6 (0.6)	4.2 (0.6)
$16^3 \times 4$	$(0.15, 1.5)$	0.115	0.69	188.5	6.0 (0.6)	8.7 (0.7)
$16^3 \times 2$	$(0.15, 1.5)$	0.115	1.38	1262.3	8.7 (0.5)	22.2 (0.7)

→ **In all cases where the interactions are non-negligible, the predictions deteriorate as T/m increases**



Backup: *Spectral properties from QCD data*

Goal: Extract information about the finite T spectral function $\rho_\Gamma(\omega,\boldsymbol{p})$ from data of *Euclidean* correlator $C_\Gamma(\tau,\vec{x}) = \langle O_\Gamma(\tau,\vec{x})\, O_\Gamma(0,\vec{0})\rangle_T$ O_Γ = scalar operator

- Standard approach: extract $\rho_\Gamma(\omega,\boldsymbol{p})$ from temporal correlator $\widetilde{C}_\Gamma(\tau,\boldsymbol{p})$

$$\widetilde{C}_\Gamma(\tau,\vec{p}) = \int_0^\infty \frac{d\omega}{2\pi} \frac{\cosh\left[\left(\frac{\beta}{2} - |\tau|\right)\omega\right]}{\sinh\left(\frac{\beta}{2}\omega\right)} \rho_\Gamma(\omega,\vec{p})$$

→ *Problem is ill-conditioned, need more information!*

- Instead, one can use the ***spatial*** correlator, where one integrates $C_\Gamma(\tau,\boldsymbol{x})$ over $\{\tau,x,y\}$ and fixes a spatial direction z

$$C_\Gamma(z) = \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} e^{ip_z z} \int_0^\infty \frac{d\omega}{\pi\omega}\, \rho_\Gamma(\omega, p_x = p_y = 0, p_z)$$

- It turns out that *if* thermoparticles exist, then they will give a distinct contribution to $C(z)$

$$C(z) \approx \frac{1}{2}\int_{|z|}^{\infty} dR\; e^{-mR} D_{m,\beta}(R)$$

[P.L., *PRD* 106 (2022); P.L., O. Philipsen, *JHEP* 10, 161 (2022)]

→ This component can be extracted *directly* from data

Backup: *Spectral properties from QCD data*

- But what impact do **thermoparticles** have on the behaviour of $C(z)$?

  → They give contributions that *dominate* for large z

  $$C(z) \approx \frac{1}{2}\int_{|z|}^{\infty} dR\; e^{-mR} D_{m,\beta}(R)$$

- If they exist, one can extract their damping factors $D_\beta(\boldsymbol{x})$ *directly* from $C(z)$ and then determine the spectral contribution $\rho_{\mathrm{TP}}(\omega,\boldsymbol{p})$

$C(z)$ data → $D_\beta(\boldsymbol{x})$

\+

$$\rho(\omega,\vec{p}) = \int_0^\infty ds \int \frac{d^3\vec{u}}{(2\pi)^2}\, \epsilon(\omega)\, \delta\big(\omega^2 - (\vec{p}-\vec{u})^2 - s\big)\, \widetilde{D}_\beta(\vec{u},s)$$

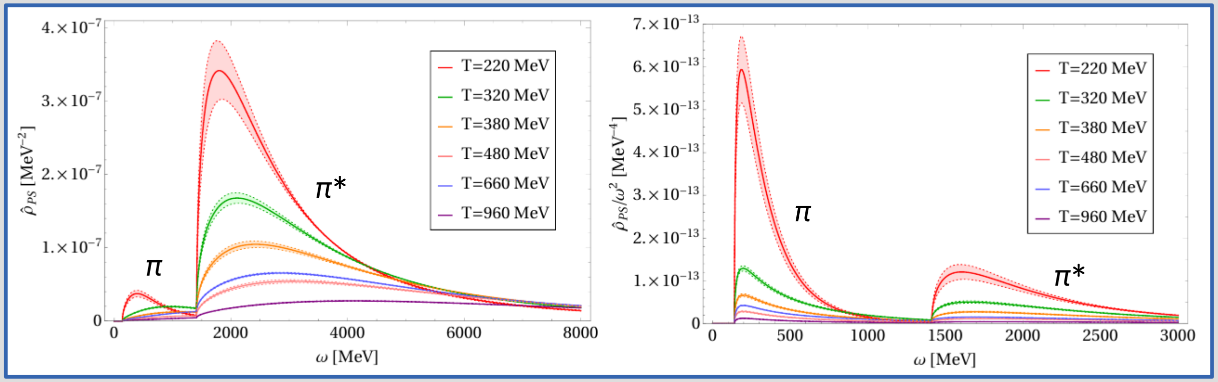
} → $\rho_{\mathrm{TP}}(\omega,\boldsymbol{p})$

- Apply approach to QCD lattice data → simple case is the spatial correlator $C_{\mathrm{PS}}(z)$ of the pseudo-scalar meson operator $\mathcal{O}^a_{\mathrm{PS}} = \overline{\psi}\gamma_5\frac{\tau^a}{2}\psi$

→ Done for light-light (pion), light-strange (kaon), and strange-strange (eta) quarks [P.L., O. Philipsen, 2022; D. Bala, O. Kaczmarek, P.L., O. Philipsen, and T. Ueding, 2310.13476]

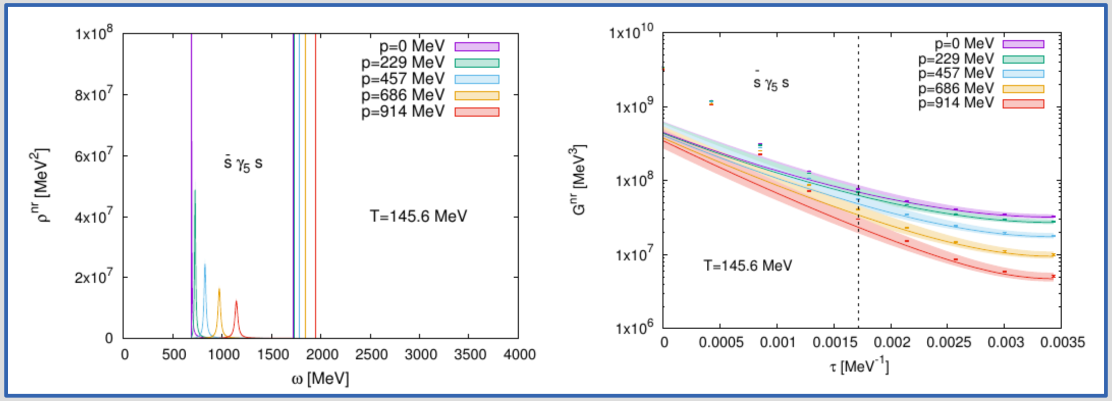
Backup: *Spectral properties from QCD data*

- Apply approach to lattice QCD data → simple case is the spatial correlator $C_{\mathrm{PS}}(z)$ of the pseudo-scalar meson operator $\boxed{\mathcal{O}^a_{\mathrm{PS}} = \overline{\psi}\gamma_5 \frac{\tau^a}{2}\psi}$
- Analysis performed for various pseudo-scalar meson operators:



Light-light pseudo-scalar meson (pion) channel [P.L., O. Philipsen, *JHEP* 10, 161 (2022)]

Light-strange (kaon) and strange-strange (eta) pseudo-scalar meson channels [D. Bala, O. Kaczmarek, P. L., O. Philipsen, and T. Ueding, *JHEP* 05, 332 (2024)]



Data in *all* channels consistent with a thermoparticle-type ground state: suggests light pseudo-scalar mesons (pions, kaons,..) still have a bound-state-like structure, even at high T

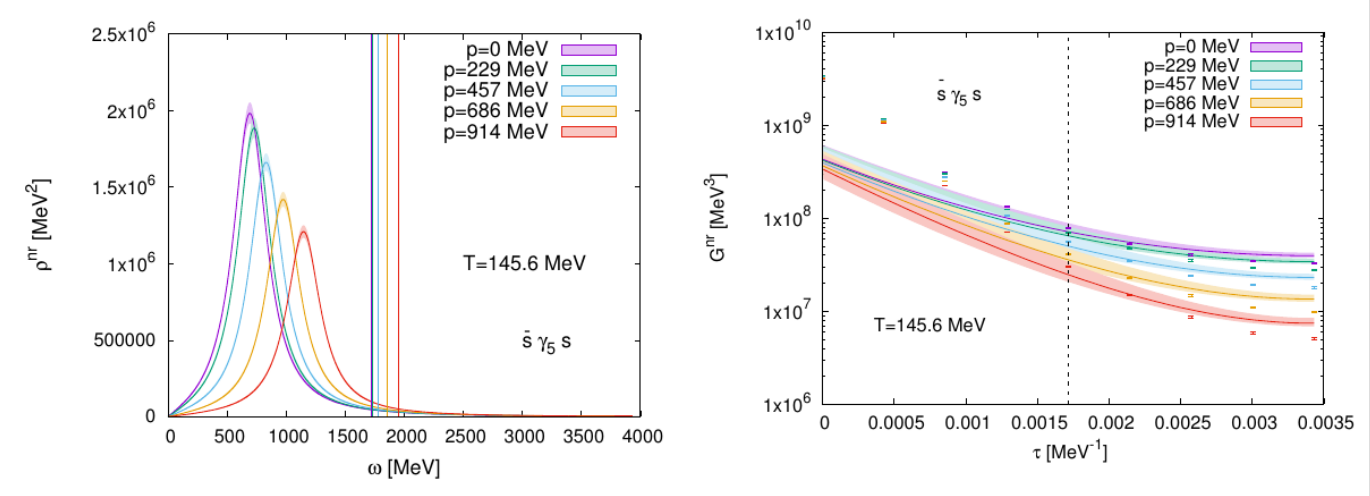
Backup: *Spectral properties from QCD data*

- The robustness of the thermoparticle hypothesis can also be tested by comparing with different causal models, e.g. a Breit Wigner

$$\rho_{\rm BW}(\omega,\vec{p}) = \frac{4\omega\Gamma}{(\omega^2-|\vec{p}|^2-m^2-\Gamma^2)^2+4\omega^2\Gamma^2},$$

$$C_{\rm BW}(z) = \frac{e^{-\sqrt{m^2+\Gamma^2}|z|}}{2\sqrt{m^2+\Gamma^2}}.$$

- Same procedure as with the thermoparticle case: (i) extract the width parameter Γ and coefficient from the spatial lattice data (ii) use this to predict the corresponding temporal correlator



→ Data is *not* consistent with a Breit-Wigner-type ground state!

Backup: *An alternative approach*

→ *But given a specific QFT, what form should these components take?*

Idea: thermal scattering states are defined by imposing an asymptotic field condition [Bros, Buchholz, (2002)]:

Asymptotic fields Φ_0 are assumed to satisfy dynamical equations, but only at large x_0

In Φ^4 theory

$$(\partial^2 + m^2)\phi_0(x) + \frac{\lambda}{3!}\phi_0^3(x) \xrightarrow{|x_0|\to\infty} 0$$

- Since thermoparticles dominate the large-time behaviour of correlators, they are natural candidates for describing such states. It turns out that their damping factors $\widetilde{D}_{m,\beta}(\boldsymbol{u})$ are **uniquely fixed** by the asymptotic condition
- In Φ^4 theory one finds (where κ is a thermal width):

For $g < 0$:

$$\widetilde{D}^{(-)}_{m,\beta}(\vec{u}) = \frac{2\pi^2}{\kappa^2}\delta(|\vec{u}| - \kappa),$$

↓

$$\widetilde{G}^{(-)}_{\beta}(k_0, \vec{p}) = \frac{1}{4|\vec{p}|\kappa}\ln\left[\frac{-k_0^2 + m^2 + (|\vec{p}| + \kappa)^2}{-k_0^2 + m^2 + (|\vec{p}| - \kappa)^2}\right]$$

For $g > 0$:

$$\widetilde{D}^{(+)}_{m,\beta}(\vec{u}) = \frac{4\pi}{\kappa_0\left(|\vec{u}|^2 + \kappa^2\right)},$$

↓

$$\widetilde{G}^{(+)}_{\beta}(k_0, \vec{p}) = \frac{i}{2|\vec{p}|\kappa_0}\ln\left[\frac{\sqrt{-k_0^2 + m^2} - i|\vec{p}| + \kappa}{\sqrt{-k_0^2 + m^2} + i|\vec{p}| + \kappa}\right]$$