Correlations and fluctuations in a thermal system with non-abelian charges

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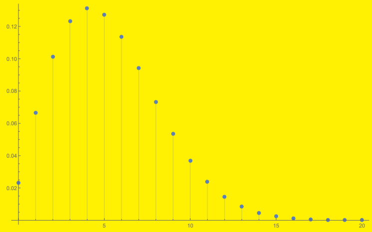
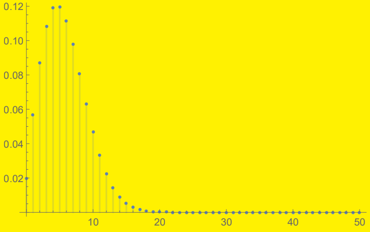
... are not quite true, as e.g.

- Uniform mixture of π^+, π^-, and π^0 would prefer to be in the isosinglet state.
- Uniform mixture of protons and neutrons would prefer to be in the isosinglet state

$\sharp\pi^+ = \sharp\pi^0 = \sharp\pi^- = 15$



$\sharp$ protons $= \sharp$ neutrons $=50$



- These are expansions of these states into corresponding irreducible representations $SU(2)$.

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Statistical ensembles of high energy physics

The thermodynamic system of volume V and temperature T composed of charged particles and their antiparticles carrying charge ± 1.
The partition functions of the canonical and grand canonical statistical system

$$\mathcal{Z}_Q^C(V,T) = \mathrm{Tr}_Q\, \mathrm{e}^{-\beta \hat{H}} = \sum_{N_+ - N_- = Q}^{\infty} \frac{z^{N_- + N_+}}{N_-!N_+!} = I_Q(2Vz_0)\,,$$

$$\mathcal{Z}^{GC}(V,T) = \mathrm{Tr}\, \mathrm{e}^{-\beta(\hat{H} - \mu \hat{Q})} = \exp\left(2Vz_0 \cosh\frac{\mu}{T}\right).$$

Vz_0 is the sum over all one-particle partition functions

$$z_0^{(i)}(T) = \frac{1}{V}\frac{V}{(2\pi)^3} g_i \int d^3p\, e^{-\beta\sqrt{p^2+m_i^2}} = \frac{1}{2\pi^2} T g_i\, m_i^2\, K_2\left(\frac{m_i}{T}\right),$$

g_i – the spin degeneracy factor.

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Abelian and nonabelian

Example

Perform and compare results of the statistical system: nucleons (n, p) and pions $(\pi^{\pm}, \pi^0)$ with an exact isospin $SU(2)$ and $U(1)_B$ symmetry.

- Abelian approach based on $U(1)_{I_3} \times U(1)_B$ symmetry. Abelian canonical partition function is given as

$$\mathcal{Z}^{(a)}_{B,I_3} = \mathrm{Tr}_{B,I_3}\, \mathrm{e}^{-\beta H}$$

  with the trace-sum over all states with the given value I_3 of the third component of the isospin.

- Nonabelian approach based $SU(2) \times U(1)_B$ symmetry. Nonabelian canonical partition function is given as

$$\mathcal{Z}^{(na)}_{B,I} = \mathrm{Tr}_{B,I}\, \mathrm{e}^{-\beta H}$$

  with the trace-sum over all states with the given value I of the total isospin.

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General projective approach

A generating function is given as

$$\tilde{\mathcal{Z}}(g) = \mathrm{Tr}\{U(g)\,\mathrm{e}^{-\beta H}\} = \sum_{\Lambda} \frac{\chi_\Lambda(g)}{dim(\Lambda)}\, Z_\Lambda^{(na)}$$

$$Z_\Lambda^{(na)} = \mathrm{Tr}_\Lambda\, \mathrm{e}^{-\beta H} \ .$$

Then

$$Z_\Lambda^{(na)} = dim(\Lambda) \int d\mu(g)\, \chi_\Lambda(g) \tilde{\mathcal{Z}}(g)\,.$$

Technical details

Redlich K., and T.L.: Z. Phys. **C 5** (1980) 201
T.L.: Phys. Lett. **B 104** (1981) 153

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Thermal models calculations - in principle

$$\epsilon = \frac{1}{2\pi^2}\sum_{i=1}^{l}(2s_i+1)\int_0^\infty dp\,\frac{p^2 E_i}{\exp\left\{\frac{E_i-\mu_i}{T}\right\}\pm 1}\,,$$

$$n_B = \frac{1}{2\pi^2}\sum_{i=1}^{l}(2s_i+1)\int_0^\infty dp\,\frac{p^2 B_i}{\exp\left\{\frac{E_i-\mu_i}{T}\right\}\pm 1}\,,$$

$$n_S = \frac{1}{2\pi^2}\sum_{i=1}^{l}(2s_i+1)\int_0^\infty dp\,\frac{p^2 S_i}{\exp\left\{\frac{E_i-\mu_i}{T}\right\}\pm 1}\,,$$

$$n_Q = \frac{1}{2\pi^2}\sum_{i=1}^{l}(2s_i+1)\int_0^\infty dp\,\frac{p^2 Q_i}{\exp\left\{\frac{E_i-\mu_i}{T}\right\}\pm 1}\,.$$

where

$$\mu_j = b_j\mu_b + s_j\mu_s + +q_j\mu_q$$

$SU(2)$ case

One can compare analytically abelian and nonabelian approach
Characters if irreducible representation are given as

$$\chi_J(\gamma) = \frac{\sin\left(J + \frac{1}{2}\right)\gamma}{\sin\frac{\gamma}{2}} = \sum_{j_3=-J}^{J} e^{ij_3\gamma}$$

with the measure

$$d\mu(\gamma) = \sin^2\frac{\gamma}{2}\,d\gamma = \frac{1-\cos\gamma}{2}\,d\gamma$$

and the integration domain $\{0, 2\pi\}$.

Projections

A generating function is given as

$$\tilde{\mathcal{Z}} = \mathrm{Tr}\{U(g)\,\mathrm{e}^{-\beta H}\} = \sum_{J=0}^{\infty} \frac{\chi_J(\gamma)}{2J+1}\, Z_J^{(na)}\,; \qquad Z_J^{(na)} = \mathrm{Tr}_J\, \mathrm{e}^{-\beta H}\ .$$

So we have

$$Z_J^{(na)} = \frac{2J+1}{\pi} \int_0^{2\pi} d\gamma\, \chi_J(\gamma) \tilde{\mathcal{Z}}(\gamma) \sin^2 \frac{\gamma}{2}\,.$$

The abelian canonical partition function

$$Z_{j_3}^{(a)} = \mathrm{Tr}_{j_3}\, \mathrm{e}^{-\beta H} = \frac{1}{2\pi} \int_0^{2\pi} d\gamma\, \tilde{\mathcal{Z}}(\gamma)\, \mathrm{e}^{-i j_3 \gamma} = \frac{1}{2\pi} \int_0^{2\pi} d\gamma\, \tilde{\mathcal{Z}}(\gamma)\, \cos j_3 \gamma\,.$$

Projections from trigonometry

For the abelian canonical partition function $Z_{j_3}^{(a)}$

$$Z_{j_3}^{(a)} = \mathrm{Tr}_{j_3}\, \mathrm{e}^{-\beta H} = \frac{1}{2\pi}\int_0^{2\pi} d\gamma\, \tilde{\mathcal{Z}}(\gamma)\, \mathrm{e}^{-ij_3\gamma} = \frac{1}{2\pi}\int_0^{2\pi} d\gamma\, \tilde{\mathcal{Z}}(\gamma)\, \cos j_3\gamma\,.$$

But

$$\chi_J(\gamma)\, \sin^2\frac{\gamma}{2} = \frac{\sin\left(J+\frac{1}{2}\right)\gamma}{\sin\frac{\gamma}{2}}\sin^2\frac{\gamma}{2} = \frac{1}{2}\left(\cos J\gamma - \cos\left(J+1\right)\gamma\right)\,.$$

This allows to express a nonabelian $SU(2)$ partition function by means of abelian partition functions

$$Z_J^{(na)} = (2J+1)\left(Z_J^{(a)} - Z_{J+1}^{(a)}\right)\,.$$

Direct variables

The chemical potential μ determines the average charge in the grand canonical ensemble

$$\langle Q \rangle = T \frac{\partial}{\partial \mu} \ln \mathcal{Z}^{GC} .$$

This allows to eliminate the chemical potential from further formulae for the grand canonical probabilities distributions

$$\frac{\mu}{T} = \operatorname{arcsinh} \frac{\langle Q \rangle}{2Vz_0} = \ln \frac{\langle Q \rangle + \sqrt{\langle Q \rangle^2 + 4(Vz_0)^2}}{2Vz_0} .$$

Probabilities in ensembles

To have N_- negative particles in the canonical ensemble

$$\mathcal{P}_Q^C(N_-, V) = \frac{(Vz_0)^{2N_-+Q}}{N_-!(N_- + Q)!} \frac{1}{I_Q(2Vz_0)} .$$

To have N_- negative particles in the grand canonical ensemble

$$\mathcal{P}_{\langle Q \rangle}^{GC}(N_-, V) =$$

$$\frac{1}{N_-!} \left[\frac{2(Vz_0)^2}{\langle Q \rangle + \sqrt{\langle Q \rangle^2 + 4(Vz_0)^2}} \right]^{N_-} \exp \left[- \frac{2(Vz_0)^2}{\langle Q \rangle + \sqrt{\langle Q \rangle^2 + 4(Vz_0)^2}} \right]$$

Technical details

Cleymans J., Redlich K., and Turko L. Phys. Rev. C **71** 047902 (2005)
Cleymans J., Redlich K., and Turko L. J. Phys. G **31** 1421 (2005)

For the pion gas

$$Z_{I_3} = \exp\left[\lambda_0 Z_\pi^{(1)}\right] \left(\frac{\lambda_+}{\lambda_-}\right)^{I_3/2} I_{I_3}(2Z_\pi^{(1)}\sqrt{\lambda_+\lambda_-}\,) \tag{1}$$

For the $\pi - N$ system

$$Z_{B,I_3} = \exp\left[\lambda_0 Z_\pi^{(1)}\right] \sum_{n=-\infty}^{\infty} \left(\frac{\lambda_p}{\lambda_{\bar{p}}}\right)^{n/2} \left(\frac{\lambda_n}{\lambda_{\bar{n}}}\right)^{(B-n)/2} \left(\frac{\lambda_+}{\lambda_-}\right)^{(B/2+I_3-n)/2} \times I_n(2Z_N^{(1)}\sqrt{\lambda_p\lambda_{\bar{p}}}\,) I_{B-n}(2Z_N^{(1)}\sqrt{\lambda_n\lambda_{\bar{n}}}\,) I_{B/2+I_3-n}(2Z_\pi^{(1)}\sqrt{\lambda_+\lambda_-}\,)$$

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Conclusions

- In the thermodynamic limit relevant probabilities are density distributions.
- Density probability distributions obtained from different statistical ensembles have the same thermodynamical limit.
- Finite volume effect more relevant for higher moments.
- Canonical suppression factor for particles depends on densities.
- Canonical ensembles based on the nonabelian symmetries are different from ensembles based on the direct product of abelian subgroups.
- Quantitative results are also different.
- There is a hope to calculate canonical "nonabelian" partition function without using poorly defined oscilating integrals - also for higher internal symmetries, beyond $SU(2)$.
- ...work in progress.

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