Four types of phase transitions in interacting meson (boson) matter at high temperatures

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Outline

- Introduction
- $\varphi^4 + \varphi^6$ model at $n_I = 0$
- Creation of the Bose-Einstein condensate: phase transitions of the 2nd and 1st order
- Thermodynamic mean-field model for a particle-antiparticle system
- Conserved isospin density: Four types of phase transitions
- Concluding remarks

High energy collisions

N

P

E ~ 1 TeV

P   P

π - mesons

Other particles
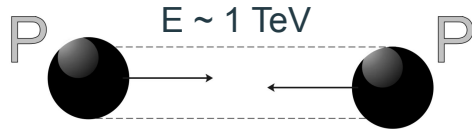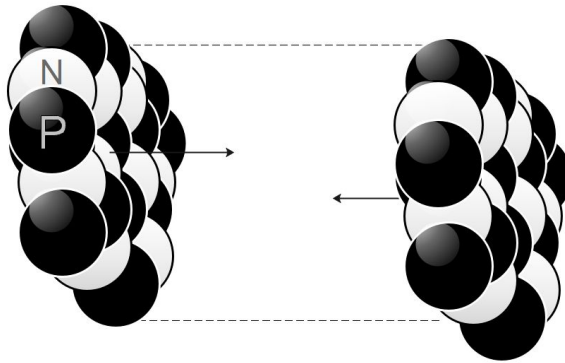
K - mesons

$N \sim 10^2 - 10^4$

Task: Study of the conditions for the occurrence of Bose condensate in the pion-antipion system with interaction
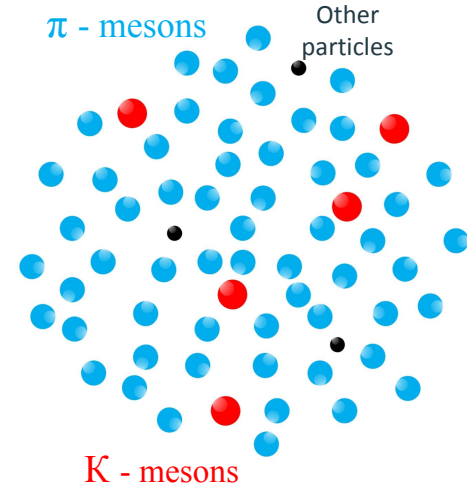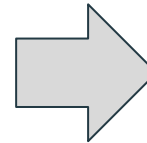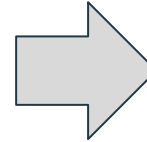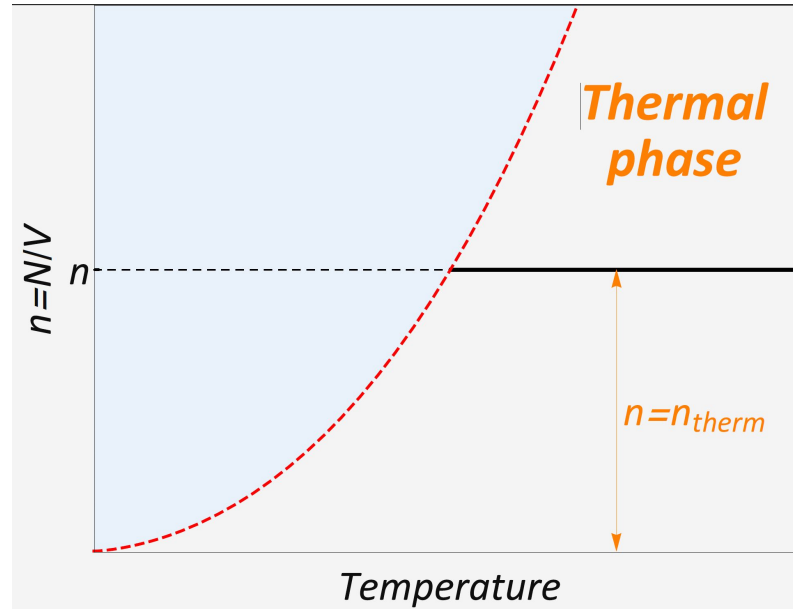
Motivation

Knowledge of the phase structure of meson systems in the regime of finite temperatures and isospin densities is crucial for understanding a wide range of phenomena, from nucleus-nucleus collisions to neutron stars, as well as cosmology. This field is an important part of hot and dense hadronic matter research. At the same time, the study of meson systems has its own specificity due to the possibility of the Bose-Einstein condensation of bosonic particles.

Introduction
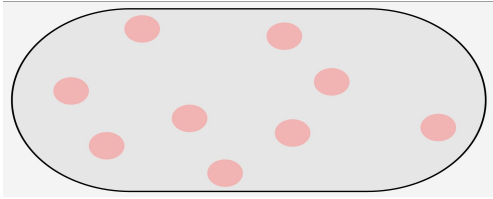
$$n = \int \frac{d^3k}{(2\pi\hbar)^3} \frac{1}{\exp\left[\frac{k^2/2m - \mu}{T}\right] - 1}$$

$$f_{\text{BE}}(T, \mu) = \frac{1}{\exp\left[\frac{k^2/2m - \mu}{T}\right] - 1}$$

Thermal phase

n

$n = n_{therm}$

n=N/V

Temperature

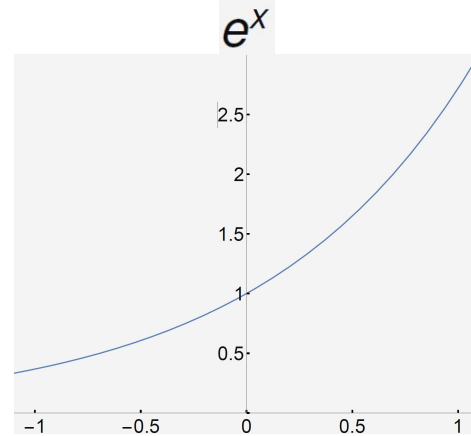# Limitation on chemical potential

$$f_{BE}(x) = \frac{1}{e^x - 1}$$

$$f_{BE}(E_j, \mu) = \frac{1}{\exp\left[\frac{E_j - \mu}{T}\right] - 1}$$

$$E_0 < E_1 < E_2 < \ldots \quad \rightarrow \quad E_0 - \mu \geq 0$$

$$\implies \quad \mu \leq E_0$$

e^x

Introduction
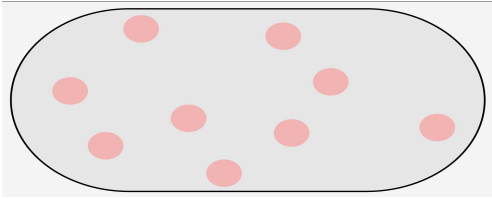
$$\frac{N}{V} = n_{\text{cond}} + \int \frac{d^3k}{(2\pi\hbar)^3} \frac{1}{\exp\left[\frac{k^2/2m - \mu}{T}\right] - 1}\Bigg|_{\mu=0}$$

$$f_{\text{BE}}(T, \mu) = \frac{1}{\exp\left[\frac{k^2/2m - \mu}{T}\right] - 1}$$
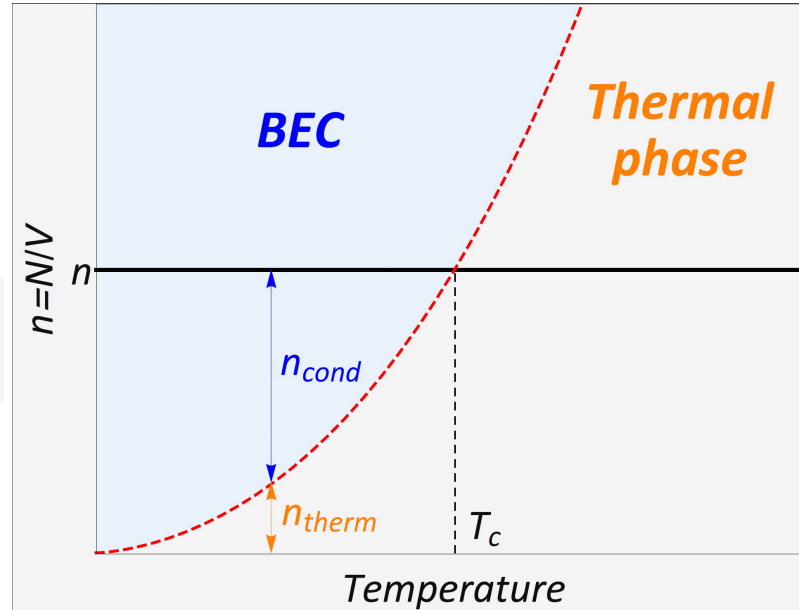
$$k_{\text{B}}T \quad \to \quad T$$
$$1\,eV = 1.16 \cdot 10^4 \text{ K}$$

Original papers of Bose and Einstein

Bose S.N., Plancks Gesetz und Lichtquantenhypothese,

Zeitschrift für Physik, 26, 178-181, (1924).

Zusammenfassung

Der Phasenraum eines Lichtquants in bezug auf ein gegebenes Volumen wird in „Zellen" von der Größe h^3 aufgeteilt. Die Zahl der möglichen Verteilungen der Lichtquanten einer makroskopisch definierten Strahlung unter diese Zellen liefert die Entropie und damit alle thermodynamischen Eigenschaften der Strahlung.

Summary

The phase space of a light quantum in relation to a given volume is divided into "cells" of size h^3. The number of possible distributions of the light quanta of a macroscopically defined radiation among these cells provides the entropy and thus all thermodynamic properties of the radiation.

Plancks Gesetz und Lichtquantenhypothese.

Von **Bose** (Dacca-University, Indien).

(Eingegangen am 2. Juli 1924.)

Der Phasenraum eines Lichtquants in bezug auf ein gegebenes Volumen wird in „Zellen" von der Größe h^3 aufgeteilt. Die Zahl der möglichen Verteilungen der Lichtquanten einer makroskopisch definierten Strahlung unter diese Zellen liefert die Entropie und damit alle thermodynamischen Eigenschaften der Strahlung.

Plancks Formel für die Verteilung der Energie in der Strahlung des schwarzen Körpers bildet den Ausgangspunkt für die Quantentheorie, welche in den letzten 20 Jahren entwickelt worden ist und in allen Gebieten der Physik reiche Früchte getragen hat. Seit der Publikation im Jahre 1901 sind viele Arten der Ableitung dieses Gesetzes vorgeschlagen worden. Es ist anerkannt, daß die fundamentalen Voraussetzungen der Quantentheorie unvereinbar sind mit den Gesetzen der klassischen Elektrodynamik. Alle bisherigen Ableitungen machen Gebrauch von der Relation

$$\varrho_\nu \, d\nu = \frac{8\,\pi\,\nu^2\,d\nu}{c^3}\,E,$$

d. h. von der Relation zwischen der Strahlungsdichte und der mittleren Energie eines Oszillators, und sie machen Annahmen über die Zahl der Freiheitsgrade des Äthers, wie sie in obige Gleichung eingeht (erster Faktor der rechten Seite). Dieser Faktor konnte jedoch nur aus der klassischen Theorie hergeleitet werden. Dies ist der unbefriedigende Punkt in allen Ableitungen, und es kann nicht wundernehmen, daß immer wieder Anstrengungen gemacht werden, eine Ableitung zu geben, die von diesem logischen Fehler frei ist.

Eine bemerkenswert elegante Ableitung ist von Einstein angegeben worden. Dieser hat den logischen Mangel aller bisherigen Ableitungen erkannt und versucht, die Formel unabhängig von der klassischen Theorie zu deduzieren. Von sehr einfachen Annahmen über den Energieaustausch zwischen Molekülen und Strahlungsfeld ausgehend, findet er die Relation

$$\varrho_\nu = \frac{\alpha_{mn}}{e^{\frac{\varepsilon_m - \varepsilon_n}{k\,T}} - 1}.$$

Original papers of Bose and Einstein

Quantentheorie des einatomigen idealen Gases (Einstein , 1924)

Einstein , 1925

So what happens when we increase the density of the substance, *n/V*, at constant temperature (for example by isothermal compression)?

I suggest that in this case, as you increase the overall density, an ever increasing number of molecules drop into the ground state (i.e. the first quantum state, state of no kinetic energy)[b], while the other molecules will distribute according to the parameter *λ=1*.

My prediction is the emergence of something very similar to what occurs when vapor is isothermally compressed beyond the saturation volume. A separation will take place; one part will "condense", the rest will remain as a "saturated ideal gas" (*A=0 λ=1*)

……………..

Hence we acquire this theorem: According to the derived state equation of ideal gases, for any given temperature there exists a maximum density of agitated molecules.

London, F. (1938a), *Nature* **141**, 643.
London, F. (1938b), *Phys. Rev.* **54**, 947.

Überreicht vom Verfasser.

SITZUNGSBERICHTE
DER PREUSSISCHEN
AKADEMIE DER WISSENSCHAFTEN.

Sitzung der physikalisch-mathematischen Klasse vom 8. Januar.

Quantentheorie des einatomigen idealen Gases.
Zweite Abhandlung.
Von A. EINSTEIN.
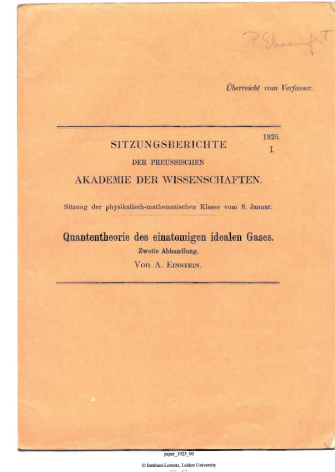
Quantentheorie des einatomigen idealen Gases.
Zweite Abhandlung.
Von A. EINSTEIN.

In einer neulich in diesen Berichten (XXII 1924, S. 261) erschienenen Abhandlung wurde unter Anwendung einer von Hrn. D. BOSE zur Ableitung der PLANCKschen Strahlungsformel erdachten Methode eine Theorie der »Entartung« idealer Gase angegeben. Das Interesse dieser Theorie liegt darin, daß sie auf die Hypothese einer weitgehenden formalen Verwandtschaft zwischen Strahlung und Gas gegründet ist. Nach dieser Theorie weicht das entartete Gas von dem Gas der mechanischen Statistik in analoger Weise ab wie die Strahlung gemäß dem PLANCKschen Gesetze von der Strahlung gemäß dem WIENschen Gesetze. Wenn die BOSEsche Ableitung der PLANCKschen Strahlungsformel ernst genommen wird, so wird man auch an dieser Theorie des idealen Gases nicht vorbeigehen dürfen; denn wenn es gerechtfertigt ist, die Strahlung als Quantengas aufzufassen, so muß die Analogie zwischen Quantengas und Molekülgas eine vollständige sein. Im folgenden sollen die früheren Überlegungen durch einige neue ergänzt werden, die mir das Interesse an dem Gegenstande zu steigern scheinen. Der Bequemlichkeit halber schreibe ich das Folgende formal als Fortsetzung der zitierten Abhandlung.

Phase transition of second-order

- P. Ehrenfest, Phasenumwandlungen im ueblichen und erweiterten Sinn, classifiziert nach dem entsprechenden Singularitaeten des thermodynamischen Potentiales, Communications from the Physical Laboratory of the University of Leiden, Supplement No. 75b (1933).

- Gregg Jaeger, The Ehrenfest Classification of Phase Transitions: Introduction and Evolution, Arch. Hist. Exact Sci. 53, 51 (1998).

- M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E.A. Cornell, Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor, Science 269, 198 (1995).

- C.C. Bradley, C.A. Sackett, J.J. Tollett, and R.G. Hulet, Evidence of Bose-Einstein Condensation in an Atomic Gas with Attractive Interactions, Phys. Rev. Lett. 75, 1687 (1995).

1-component ideal gas
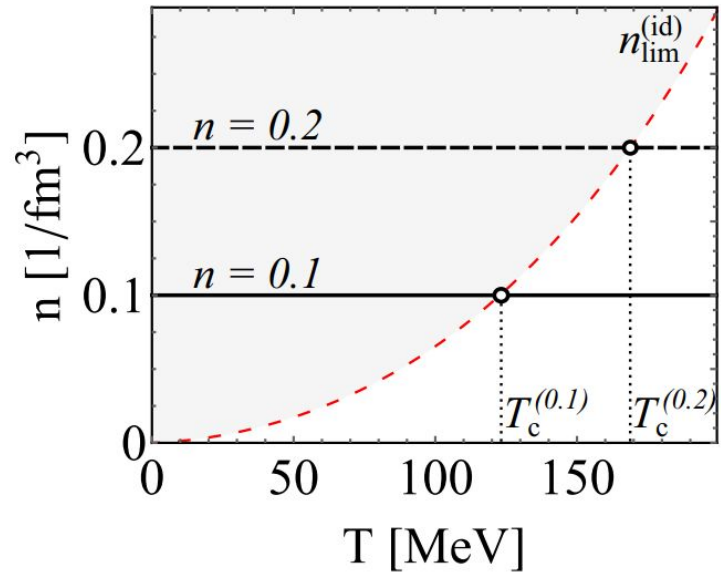
1-component ideal gas
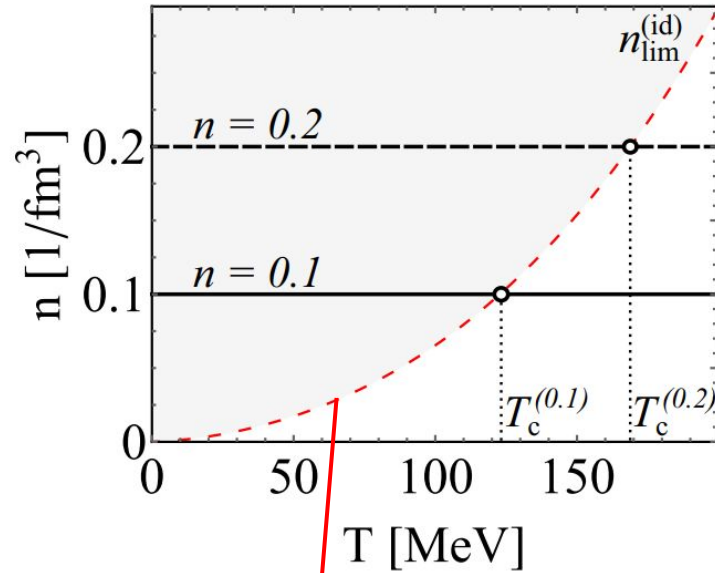
$$n_{\text{lim}}^{(\text{id})}(T) = g \int_0^\infty \frac{d^3k}{(2\pi)^3} \frac{1}{\exp\left[\frac{\sqrt{m^2+k^2}-m}{T}\right] - 1}$$

Ideal gas: particle density, energy density and heat capacity
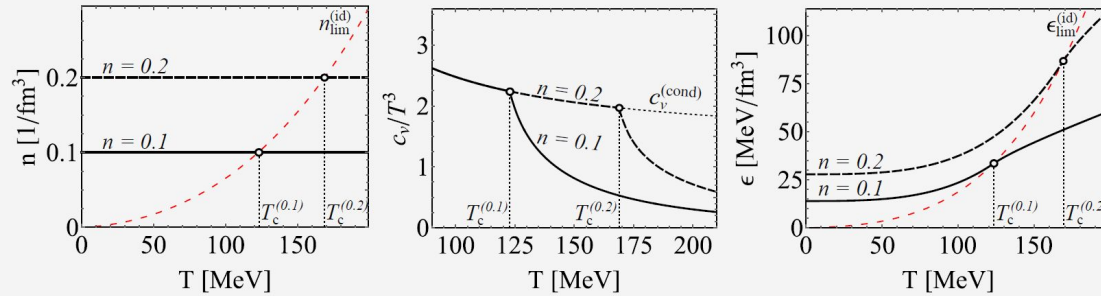
Critical curves $n_{\text{lim}}^{(\text{id})}$, $\varepsilon_{\text{lim}}^{(\text{id})}$:

$$n_{\text{lim}}^{(\text{id})}(T) = g \int \frac{d^3k}{(2\pi)^3} \frac{1}{\exp\left[\frac{\sqrt{m^2+k^2}-m}{T}\right]-1},$$

$$\varepsilon_{\text{lim}}^{(\text{id})}(T) = g \int \frac{d^3k}{(2\pi)^3} \frac{\boldsymbol{k}^2}{\sqrt{m^2+\boldsymbol{k}^2}} \frac{1}{\exp\left[\frac{\sqrt{m^2+\boldsymbol{k}^2}-m}{T}\right]-1},$$

$$c_V = \frac{\partial \varepsilon}{\partial T}\bigg|_V.$$

Condensation of interacting scalar bosons

$$\mathcal{L}(x) = \frac{1}{2}\left[\partial_\mu\hat{\phi}(x)\partial^\mu\hat{\phi}(x) - m^2\,\hat{\phi}^2(x)\right] + \mathcal{L}_{\text{int}}[\hat{\phi}^2(x)]$$

where $x = (t,\,\boldsymbol{r})$.

We adopt that:

$$\hat{\phi}(\boldsymbol{r}) = \phi_{\text{cond}} + \hat{\psi}(\boldsymbol{r}),\quad\text{where}\quad\left\langle\hat{\psi}(\boldsymbol{r})\right\rangle = 0\,.$$

Here we use the famous Bogolyubov's decomposition of the field operator into two contributions [1]

$$\Psi(\boldsymbol{r}) = \frac{1}{\sqrt{V}}a_0 + \frac{1}{\sqrt{V}}\sum_{\boldsymbol{k}\neq 0}a_k e^{i\boldsymbol{k}\cdot\boldsymbol{r}/\hbar}\,.$$

Due to the argument that at $T \to 0$ in a nonperfect Bose-Einstein gas the number of particles on the ground state

$$N_0 = \langle a_0^+ a_0\rangle \approx N\,,$$

one can treat a_0 and a_0^+ as classical values.

[1]   N. Bogolubov, *On the theory of superfluidity*, Sov. J. Phys. **11**, 23 (1947).

M.M. Bogolyubov, *Lekciyi z kvantovoyi statystyky*, Kyiv, 1947 (Ukrainian).

N.N. Bogoliubov, *Lectures on Quantum Statistics*, Gordon and Breach, New York, 1967.

Relativistic system of scalar bosons

Lagrangian density

$$\mathcal{L}(x) = \partial_\mu \hat{\phi}^+(x) \partial^\mu \hat{\phi}(x) - m^2 \hat{\sigma}(x) + \mathcal{L}_{\text{int}}[\hat{\sigma}(x)]$$

Here we use $\hat{\sigma}$ for a scalar density:

$$\hat{\sigma}(x) = \hat{\phi}^+(x) \hat{\phi}(x)$$

Quantum statistics:

$$\left\langle \hat{A} \right\rangle = \frac{1}{Z} \text{Tr}\left[e^{-\beta(\hat{H} - \mu \hat{N})} \hat{A} \right], \qquad Z = \text{Tr}\left[e^{-\beta(\hat{H} - \mu \hat{N})} \right]$$

The mean-field approximation [2]: $\sigma = \langle \hat{\sigma} \rangle$

$$\mathcal{L}_{\text{int}}(\hat{\sigma}) \simeq \mathcal{L}_{\text{int}}(\sigma) + \frac{\partial \mathcal{L}_{\text{int}}(\sigma)}{\partial \sigma}(\hat{\sigma} - \sigma)$$

[2] I. N. Mishustin, D. V. Anchishkin, L. M. Satarov, O. S. Stashko, and H. Stoecker, Phys. Rev. C 2019, 100, 022201; [arXiv:1905.09567].

$$\mathcal{L}(x) \simeq \partial_\mu \hat{\phi}^+(x) \partial^\mu \hat{\phi}(x) - M^2(\sigma) \hat{\phi}^+(x) \hat{\phi}(x) + P_{\text{ex}}(\sigma) \,,$$

Definitions of the mean field, excess pressure and effective mass:

$$U(\sigma) \equiv - \frac{\partial \mathcal{L}_{\text{int}}(\sigma)}{\partial \sigma} \,, \qquad P_{\text{ex}}(\sigma) \equiv \mathcal{L}_{\text{int}}(\sigma) - \sigma \frac{\partial \mathcal{L}_{\text{int}}(\sigma)}{\partial \sigma} \,,$$

$$M^2(\sigma) \equiv m^2 + U(\sigma)$$

Consistency relation:

$$\sigma \frac{\partial U(\sigma)}{\partial \sigma} = \frac{\partial P_{\text{ex}}(\sigma)}{\partial \sigma}$$

Mean scalar density $\sigma = \sigma_0 + \sigma_{\text{th}}$

$$\left\langle \hat{\phi}^+ \hat{\phi} \right\rangle = \left\langle |\Phi_0|^2 + \Phi_0^* \hat{\psi} + \Phi_0 \hat{\psi}^+ + \hat{\psi}^+ \hat{\psi} \right\rangle = |\Phi_0|^2 + \left\langle \hat{\psi}^+ \hat{\psi} \right\rangle$$

Hamiltonian density in the mean-field approximation

$$\hat{\pi}(x) = \partial_t \hat{\phi}^+(x), \qquad \left[\hat{\phi}(t, \boldsymbol{r}), \hat{\pi}(t, \boldsymbol{r}')\right] = i\delta^3(\boldsymbol{r} - \boldsymbol{r}')$$

Effective Hamiltonian density $\mathcal{H} = \hat{\pi}\,\partial_t\hat{\phi} - \mathcal{L}$

$$\mathcal{H} \simeq \hat{\pi}^+(x)\hat{\pi}(x) + \boldsymbol{\nabla}\hat{\phi}^+(x)\cdot\boldsymbol{\nabla}\hat{\phi}(x) + M^2(\sigma)\hat{\phi}^+(x)\hat{\phi}(x) - P_{\text{ex}}(\sigma)$$

Using solutions of the Klein-Gordon equation

$$\partial^\mu\partial_\mu\hat{\phi} + M^2(\sigma)\hat{\phi} = 0$$

one can represent the scalar field as

$$\hat{\phi}(x) = \Phi_0 + \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\boldsymbol{k}}}}\left[a_{\boldsymbol{k}}e^{-ik\cdot x} + b_{\boldsymbol{k}}^+ e^{ik\cdot x}\right]$$

where $k^0 = \omega_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 + M^2(\sigma)}$.

$$\left[a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}^+\right] = (2\pi)^3 2\omega_k\delta(\boldsymbol{k} - \boldsymbol{k}'), \qquad \left[b_{\boldsymbol{k}}, b_{\boldsymbol{k}'}^+\right] = (2\pi)^3 2\omega_k\delta(\boldsymbol{k} - \boldsymbol{k}')$$

Self-consistent equations in the thermal phase

Hamiltonian in the mean-field approximation:

$$H(\sigma) = VM(\sigma)n_{\text{cond}} + \int \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k \left[a^+(\boldsymbol{k})a(\boldsymbol{k}) + b^+(\boldsymbol{k})b(\boldsymbol{k})\right] - VP_{\text{ex}}(\sigma)$$

$$\mu^{(-)} = \mu_I, \qquad \mu^{(+)} = -\mu_I$$

$$\langle a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}'} \rangle = (2\pi)^3 2\omega_k \delta^3(\boldsymbol{k} - \boldsymbol{k}') f_{\text{BE}}(\omega_k, \mu_I)$$

$$\langle b_{\boldsymbol{k}}^+ b_{\boldsymbol{k}'} \rangle = (2\pi)^3 2\omega_k \delta^3(\boldsymbol{k} - \boldsymbol{k}') f_{\text{BE}}(\omega_k, -\mu_I)$$

where $f_{\text{BE}}(\omega, \mu_I)$ is the Bose-Einstein distribution function ($T = 1/\beta$)

$$f_{\text{BE}}(\omega, \mu_I) = \frac{1}{e^{\beta(\omega - \mu_I)} - 1}$$

I. Particles and antiparticles are completely in the thermal phase, $T > T_c$

$$\sigma = \int \frac{d^3k}{(2\pi)^3 2\omega_k(\sigma)} \left\{ f_{\text{BE}}\left[\omega_k(\sigma), \mu_I\right] + f_{\text{BE}}\left[\omega_k(\sigma), -\mu_I\right] \right\},$$

$$n_I = \int \frac{d^3k}{(2\pi)^3} \left\{ f_{\text{BE}}\left[\omega_k(\sigma), \mu_I\right] - f_{\text{BE}}\left[\omega_k(\sigma), -\mu_I\right] \right\}$$

$$n_I = 0 \quad \rightarrow \quad \mu_I = 0$$

I. Thermal phase: $T > T_c$

$$\sigma = \int \frac{d^3 k}{(2\pi)^3 \, \omega_k(\sigma)} \, f_{\mathrm{BE}} \left[\omega_k(\sigma), \mu_I = 0 \right]$$

For a given T we solve equation with respect to σ and get $\sigma(T)$.

II. Condensate phase

$$\sigma = \sigma_{\mathrm{cond}}(T) + \int \frac{d^3 k}{(2\pi)^3 \, \omega_k(\sigma)} \, f_{\mathrm{BE}} \left[\omega_k(\sigma), \mu_I = 0 \right]$$

For a given T we solve equation with respect to σ_{cond} and get $\sigma_{\mathrm{cond}}(T)$.

$$\omega_k = \sqrt{k^2 + M^2(\sigma)} \quad \rightarrow \quad E_{\mathrm{ground-state}} = M(\sigma) \quad \rightarrow \quad \mu_I = M(\sigma)$$

Condition of a condensate creation:

$$M(\sigma) = 0$$

Bosonic system with $\varphi^4 + \varphi^6$ interaction [3]

Interaction Lagrangian:

$$\mathcal{L}_{\text{int}}(\hat{\sigma}) = \frac{a}{2}\hat{\sigma}^2(x) - \frac{b}{3}\hat{\sigma}^3(x), \qquad \hat{\sigma} = \hat{\phi}^+(x)\hat{\phi}(x)$$

$$\sigma = \left\langle \hat{\phi}^+(x)\hat{\phi}(x) \right\rangle$$

Mean field and excess pressure:

$$U(\sigma) = -a\,\sigma + b\,\sigma^2, \qquad P_{\text{ex}}(\sigma) = -\frac{1}{2}a\,\sigma^2 + \frac{2}{3}b\,\sigma^3$$

Effective mass:

$$M^2(\sigma) = m^2 + U(\sigma) = m^2 - a\,\sigma + b\,\sigma^2$$

[3] D.V. Anchishkin, A.V. Nazarenko, Liquid-like phases of $\pi^+\,\pi^-$ matter, J. Phys. Studies **10**, 93 (2006); arXiv: 0611040 [nucl-th].

Condition for the formation of condensate

Effective mass:

$$M^2(\sigma) = m^2 - a\sigma + b\sigma^2$$

The condition for the occurrence of scalar condensate at $\mu_I = 0$:

$$M^2(\sigma) = m^2 - a\sigma + b\sigma^2 = 0$$

Solutions of equation:

$$\sigma_1 = \frac{m}{\sqrt{b}}\left(\kappa - \sqrt{\kappa^2 - 1}\right), \quad \sigma_2 = \frac{m}{\sqrt{b}}\left(\kappa + \sqrt{\kappa^2 - 1}\right)$$

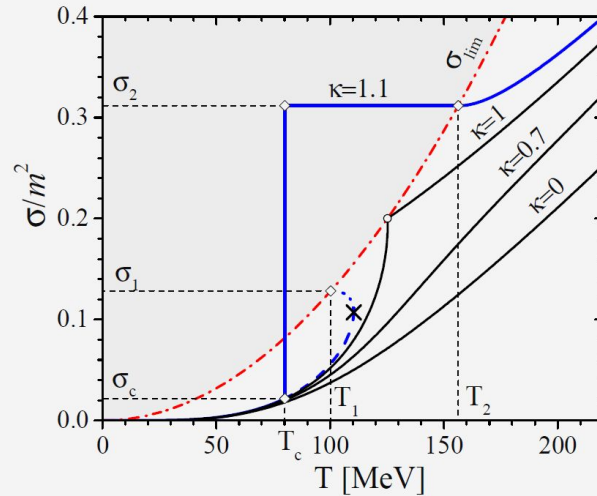To parameterize the attraction coefficient a, we introduce the dimensionless parameter κ:

$$\kappa = \frac{a}{2m\sqrt{b}} \quad \rightarrow \quad a = \kappa\, a_c, \quad a_c = 2m\sqrt{b}$$

$$m = m_\pi = 139 \text{ MeV}, \ \mu_I = 0$$

$$\sigma_{\text{lim}} = \sigma_{\text{th}}(M = 0, T) = \frac{T^2}{12}$$

- $\kappa < 1$ is the "weak" attraction,
- $\kappa > 1$ is the "strong" attraction.

$$n_I = 0 \quad \rightarrow \quad \mu_I = 0$$

II. Condensate phase

$$\sigma = \sigma_{\text{cond}}(T) + \int \frac{d^3 k}{(2\pi)^3 \, \omega_k(\sigma)} \, f_{\text{BE}}\left[\omega_k(\sigma), \mu_I = 0\right]$$

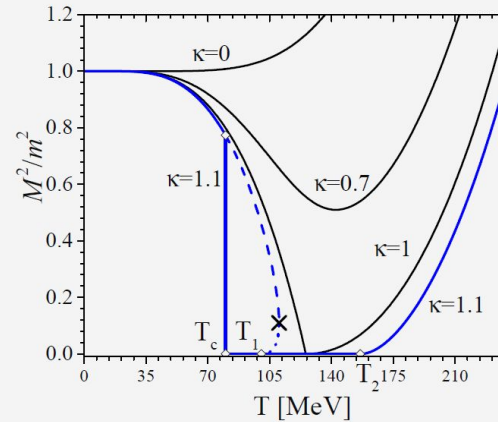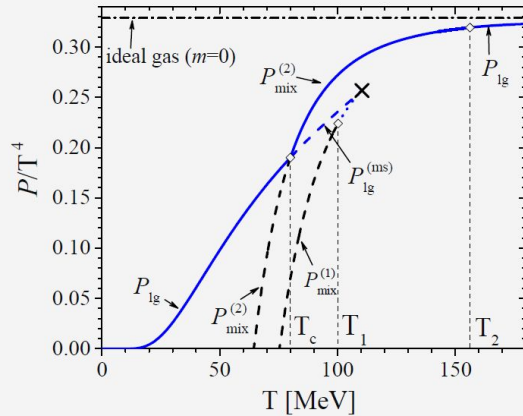Condition of a condensate creation, $\kappa > 1$:

$$M(\sigma_1) = 0, \quad M(\sigma_2) = 0, \quad \omega_{\boldsymbol{k}}(\sigma_2) = |\boldsymbol{k}|$$

$$\sigma_2 = \sigma_{\text{cond}}(T) + \int \frac{d^3 k}{(2\pi)^3 \, |\boldsymbol{k}|} \frac{1}{\exp\left(|\boldsymbol{k}|/T\right) - 1}$$

Scalar condensate: The pressure and effective mass at $\mu = 0$

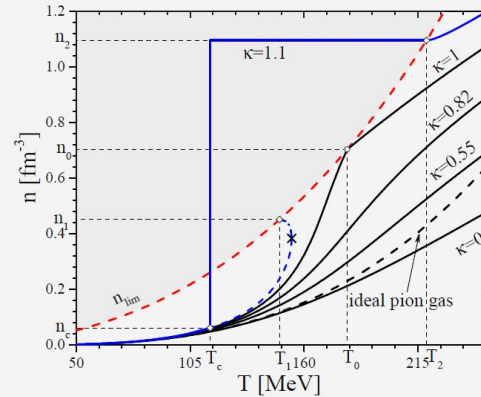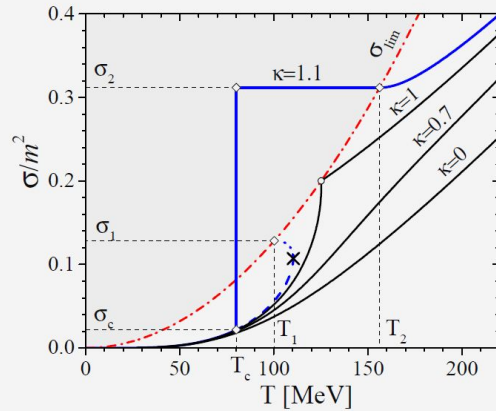$$p = \frac{\pi^2}{45} T^4 + P_{\text{ex}}(\sigma_2), \qquad M^2(\sigma) = m^2 + 2\,U(\sigma)$$

- *Left panel:* Pressure vs temperature for the supercritical case $\kappa = 1.1$. The solid blue line is the equation of state for $\kappa = 1.1$.

- *Right panel:* Effective mass vs temperature.

Comparison with thermodynamic mean-field model

$$\sigma_{\text{lim}} = \frac{T^2}{12} \qquad\qquad n_{\text{lim}} = \int \frac{d^3k}{(2\pi)^3} \left\{ \exp\left[\frac{\sqrt{m^2+k^2}-m}{T}\right] - 1 \right\}^{-1}$$

Left panel: Scalar density vs temperature $b = 25\,m_\pi^{-2}$, $a = \kappa\,2m\sqrt{b}$.

Right panel: Particle-number density vs temperature, $B = 10m_\pi\,b^2$ with $b = \frac{16\pi}{3}r_0^3$, $r_0 \approx 0.3$ fm.

[4] D. Anchishkin, V. Vovchenko, J. Phys. G **42**, 105102 (2015).

[5] D. Anchishkin, I. Mishustin, and H. Stoecker, J. Phys. G **46**, 035002 (2019).

Thermodynamic mean-field model: description of an interacting 2-component relativistic system

- P. Kerson Huang, Statistical Mechanics, John Wiley and Sons, 1987; (Section 12.3, p. 294).
  "the real conservation law deals with the conserved quantity that is the number of particles minus the number of antiparticles. That is why any study of the Bose-Einstein condensation in the relativistic Bose gas must take antiparticles into account."

- H. E. Haber and H. A. Weldon, Thermodynamics of an Ultrarelativistic Ideal Bose Gas, Phys. Rev. Lett. 46, 1497 (1981).

J. Kapusta. Bose-Einstein condensation, spontaneous symmetry breaking, and gauge theories. *Phys. Rev. D* **24**, 426 (1981) [DOI: https://doi.org/10.1103/PhysRevD.24.426].

H.E. Haber, H.A. Weldon. Finite-temperature symmetry breaking as Bose-Einstein condensation. *Phys. Rev. D* **25**, 502 (1982) [DOI: https://doi.org/10.1103/PhysRevD.25.502].

J. Bernstein, S. Dodelson. Relativistic Bose gas. *Phys. Rev. Lett.* **66**, 683 (1991) [DOI: https://doi.org/10.1103/PhysRevLett.66.683].

Thermodynamic mean-field model[1-3]

Starting point

Free energy density: $\Phi = F/V$

$$\phi(n_1, n_2, T) = \phi_1^{(0)}(n_1, T) + \phi_2^{(0)}(n_2, T) + \phi_{\text{int}}(n, T),$$

[1] D. Anchishkin and V. Vovchenko, J. Phys. G 42, 105102 (2015).
[2] D. Anchishkin, I. Mishustin, and H. Stoecker, J. Phys. G 46, 035002 (2019).
[3] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Universe 2023, 9, 411 (2023).

Thermodynamic mean-field model

Free energy density:

$$\phi(n_1, n_2, T) = \phi_1^{(0)}(n_1, T) + \phi_2^{(0)}(n_2, T) + \phi_{\text{int}}(n, T),$$

$$n = \int \frac{d^3k}{(2\pi)^3} \left[f_{\text{BE}}\big(E(k,n),\mu_I\big) + f_{\text{BE}}\big(E(k,n),-\mu_I\big) \right],$$

$$n_I = \int \frac{d^3k}{(2\pi)^3} \left[f_{\text{BE}}\big(E(k,n),\mu_I\big) - f_{\text{BE}}\big(E(k,n),-\mu_I\big) \right].$$

$$E(k,n) = \sqrt{m^2 + \boldsymbol{k}^2} + U(n)$$

$$f_{\text{BE}}(E,\mu) = \left\{ \exp\left[\frac{E-\mu}{T}\right] - 1 \right\}^{-1}$$

Thermodynamic mean-field model

Free energy density:

$$\phi(n_1, n_2, T) = \phi_1^{(0)}(n_1, T) + \phi_2^{(0)}(n_2, T) + \phi_{\text{int}}(n, T),$$

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$$E(k,n) = \sqrt{m^2 + \boldsymbol{k}^2} + U(n)$$

Fixed

$$f_{\text{BE}}(E, \mu) = \left\{ \exp\left[\frac{E - \mu}{T}\right] - 1 \right\}^{-1}$$

Thermodynamic mean-field model

Free energy density:

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$$E(k, n) = \sqrt{m^2 + \boldsymbol{k}^2} + U(n)$$

Fixed

$$f_{\text{BE}}(E, \mu) = \left\{ \exp\left[\frac{E - \mu}{T}\right] - 1 \right\}^{-1}$$

$$m = 140\,\text{MeV}$$

$$U(n) = -An + Bn^2, \quad \boxed{U(n) + m = 0} \quad n_1 = \sqrt{\frac{m}{B}}\big(\kappa - \sqrt{\kappa^2 - 1}\big), \quad n_2 = \sqrt{\frac{m}{B}}\big(\kappa + \sqrt{\kappa^2 - 1}\big),$$

$$\kappa \equiv \frac{A}{2\sqrt{mB}}.$$

Thermodynamic mean-field model

Free energy density:

$$\phi(n_1, n_2, T) = \phi_1^{(0)}(n_1, T) + \phi_2^{(0)}(n_2, T) + \phi_{\text{int}}(n, T),$$

$$n = \int \frac{d^3k}{(2\pi)^3} \left[f_{\text{BE}}\big(E(k,n), \mu_I\big) + f_{\text{BE}}\big(E(k,n), -\mu_I\big) \right],$$

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$$E(k,n) = \sqrt{m^2 + \boldsymbol{k}^2} + U(n)$$

Fixed

$$f_{\text{BE}}(E, \mu) = \left\{ \exp\left[\frac{E - \mu}{T}\right] - 1 \right\}^{-1}$$

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$$\kappa \equiv \frac{A}{2\sqrt{mB}}.$$

Interplay between A and B
Parameter B - fixed

Thermodynamic mean-field model

Free energy density:

$$\phi(n_1, n_2, T) = \phi_1^{(0)}(n_1, T) + \phi_2^{(0)}(n_2, T) + \phi_{\text{int}}(n, T),$$

$$n = \int \frac{d^3 k}{(2\pi)^3} \left[f_{\text{BE}}\big(E(k,n), \mu_I\big) + f_{\text{BE}}\big(E(k,n), -\mu_I\big) \right],$$

$$n_I = \int \frac{d^3 k}{(2\pi)^3} \left[f_{\text{BE}}\big(E(k,n), \mu_I\big) - f_{\text{BE}}\big(E(k,n), -\mu_I\big) \right].$$

$$E(k,n) = \sqrt{m^2 + \boldsymbol{k}^2} + U(n)$$

Fixed

$$f_{\text{BE}}(E, \mu) = \left\{ \exp\left[\frac{E - \mu}{T} \right] - 1 \right\}^{-1}$$

$\kappa < 1$ - weak attraction regime

$\kappa = 1$ - critical attraction regime

$\kappa > 1$ - strong(over-critical) attraction regime

$$m = 140 \, \text{MeV}$$

$$U(n) = -An + Bn^2, \quad U(n) + m = 0 \quad n_1 = \sqrt{\frac{m}{B}}\big(\kappa - \sqrt{\kappa^2 - 1}\big), \quad n_2 = \sqrt{\frac{m}{B}}\big(\kappa + \sqrt{\kappa^2 - 1}\big),$$

$$\kappa \equiv \frac{A}{2\sqrt{mB}}.$$

Interplay between A and B
Parameter B - fixed

Parametrization of $U(n)$:

$$\langle \phi^{+}\phi \rangle \;\rightarrow\; n \,, \qquad \langle \phi^{+}\phi \rangle^2 \;\rightarrow\; n^2 \,, \qquad \langle \phi^{+}\phi \rangle^3 \;\rightarrow\; n^3$$

$$U(\sigma) = -a\sigma + b\sigma^2 \;\rightarrow\; U(n) = -An + Bn^2$$

$$P_{\text{ex}}(n) = -\frac{1}{2}An^2 + \frac{2}{3}Bn^3$$

Parameterization of the attraction coefficient:

$$U(n) + m = 0$$

$$n_1 = \sqrt{\frac{m}{B}}\left(\kappa - \sqrt{\kappa^2 - 1}\right), \qquad n_2 = \sqrt{\frac{m}{B}}\left(\kappa + \sqrt{\kappa^2 - 1}\right)$$

$$\kappa \equiv \frac{A}{2\sqrt{mB}}, \quad A_{\text{c}} = 2\sqrt{mB}, \quad A = \kappa A_{\text{c}}$$

Mean field: Variations in the attraction coefficient $A = \kappa A_c$
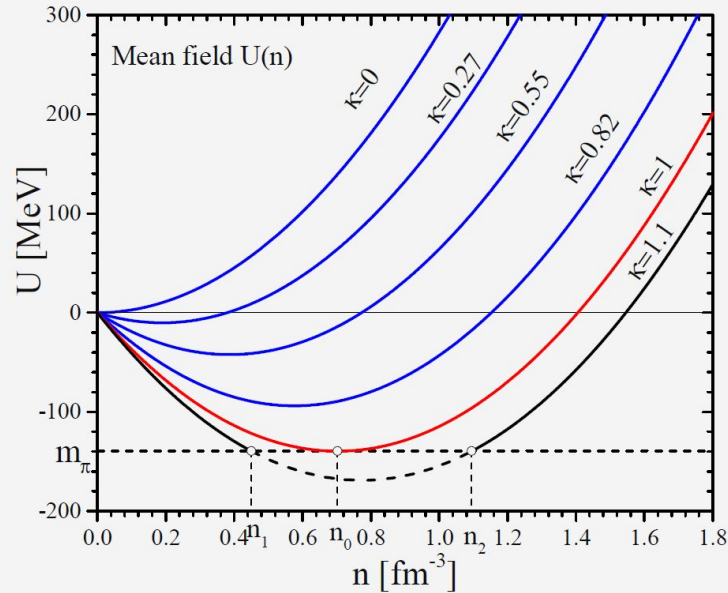
Рис.: The mean field potential U versus particle density n, $\mu = 0$. The blue curves are labeled by value of the parameter κ: $\kappa = 0$, $\kappa = 0.27$, $\kappa = 0.55$, $\kappa = 0.82$.
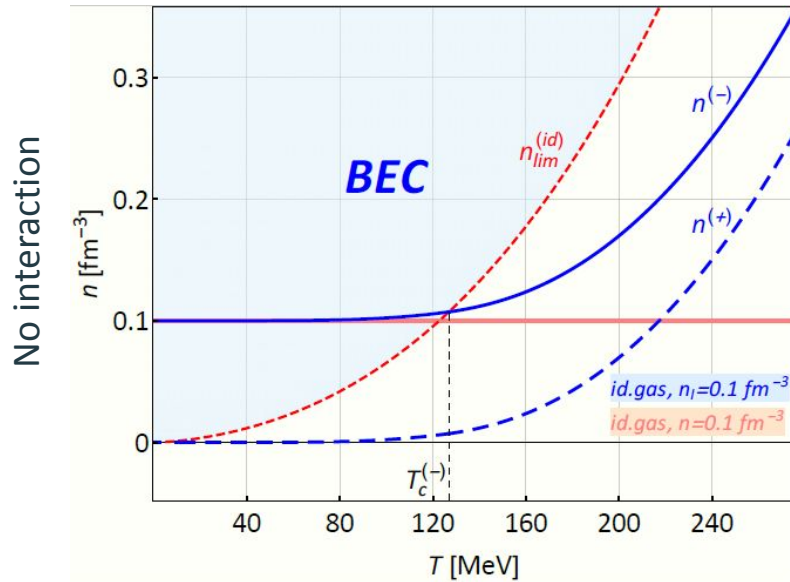
Ideal 2-component relativistic gas

Figure 1. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the ideal π^+-π^- pion gas. The total isospin density is kept constant, $n_I = 0.1\ fm^{-3}$.

$$n_I = \int \frac{d^3k}{(2\pi)^3} \left[f_{BE}(\omega_k, \mu_I) - f_{BE}(\omega_k, -\mu_I) \right]$$

$$n^{(-)} = \int \frac{d^3k}{(2\pi)^3} f_{BE}(\omega_k, \mu_I(T, n_I))$$

$$n^{(+)} = \int \frac{d^3k}{(2\pi)^3} f_{BE}(\omega_k, -\mu_I(T, n_I))$$

$$\omega_k = \sqrt{m^2 + \boldsymbol{k}^2}$$

Condition for the condensate formation:

$$\mu_I = m \qquad \text{In the interval} \quad 0 < T < Tc$$

Let's turn ON the interaction in the bosonic system!
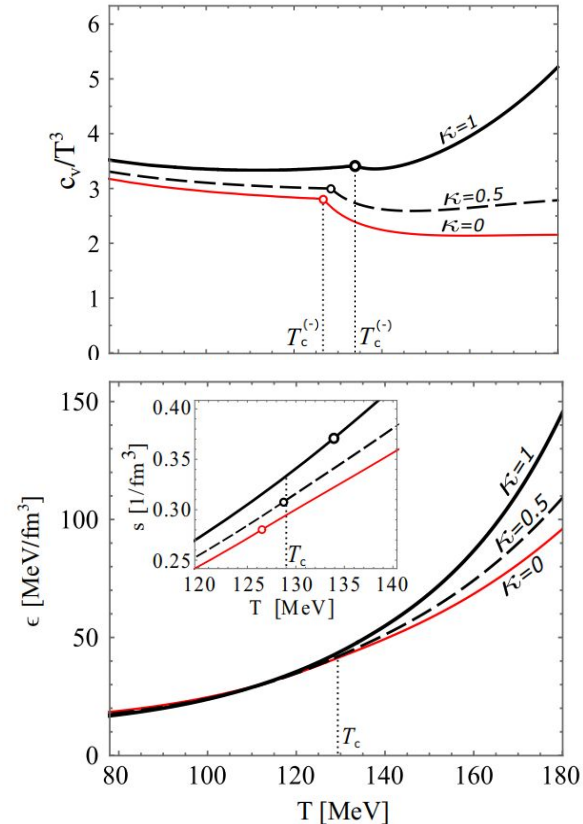
Type 1 phase transition (2nd order ph. tr.)
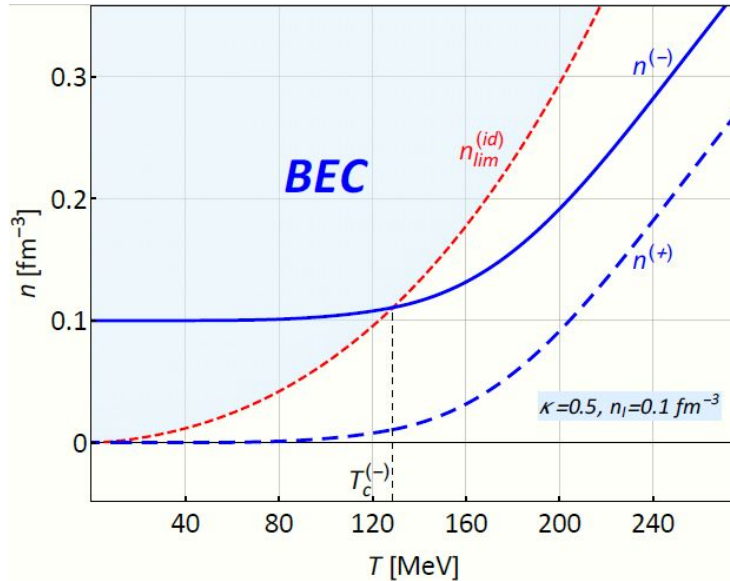
Figure 2. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas in the mean-field model. The total isospin density is kept constant, $n_I = 0.1$ fm^{-3}, and the attraction parameter is **κ = 0.5**.

Weak attraction regime
$\kappa = 0.5$

Type 1 phase transition (2nd order ph. tr.)

Figure 2. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas in the mean-field model. The total isospin density is kept constant, $n_I = 0.1$ fm^{-3}, and the attraction parameter is **$\kappa = 0.5$**.
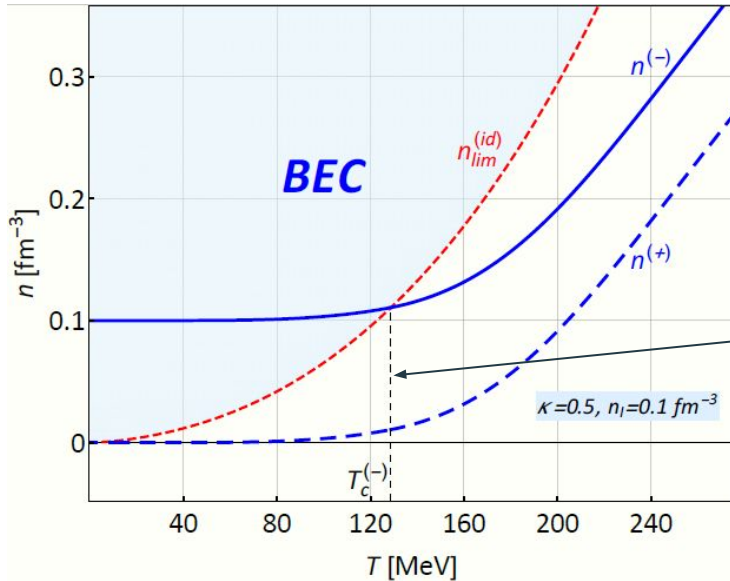
Weak attraction regime
$\kappa = 0.5$

Phase transition of the 2nd order

Type 1 phase transition (2nd order ph. tr.)
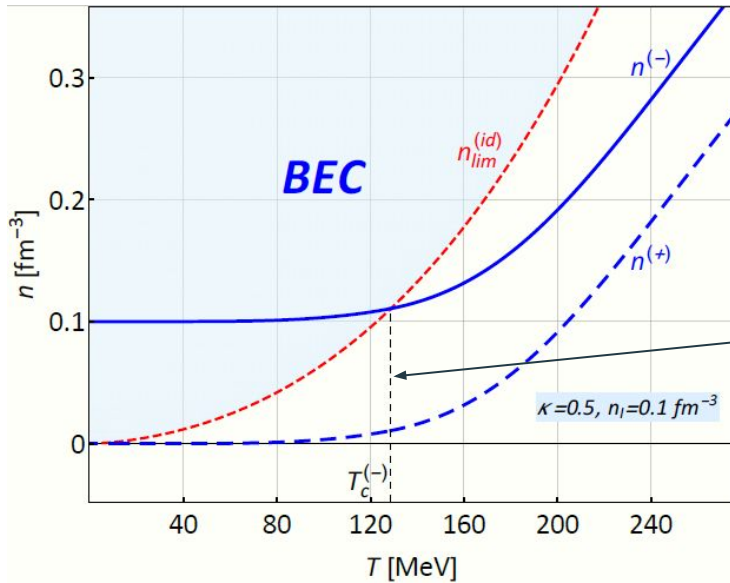
Figure 2. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas in the mean-field model. The total isospin density is kept constant, $n_I = 0.1$ fm^{-3}, and the attraction parameter is **κ = 0.5**.

Phase transition of the 2nd order

What will happen if we increase the attraction parameter κ in such a system?

Type 2 phase transition (2nd order ph. tr.)
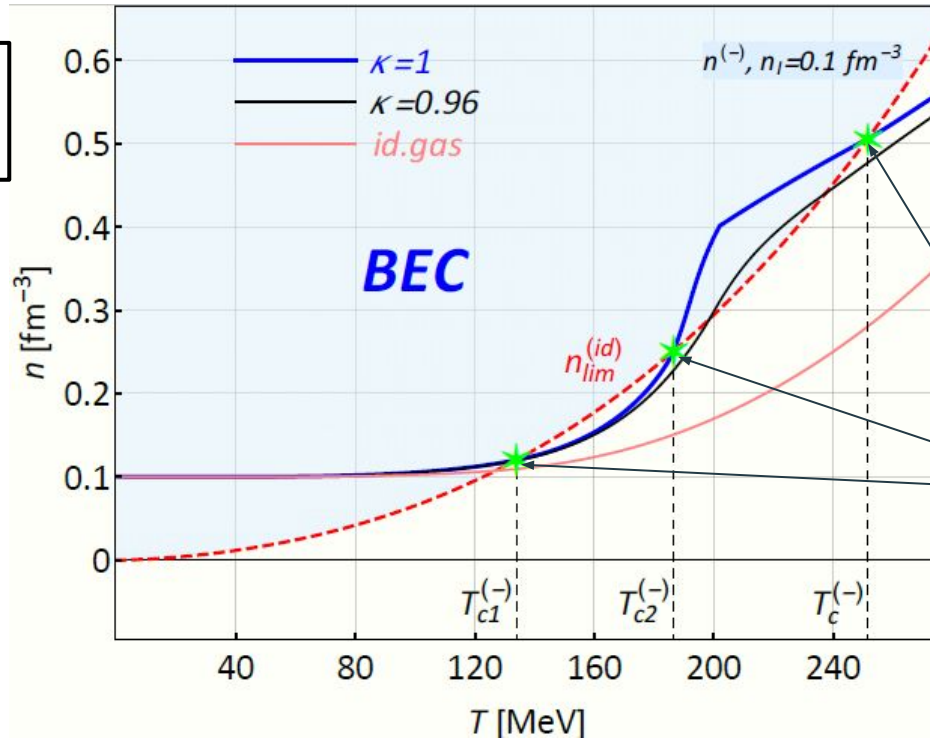
Only particles π⁻ are depicted
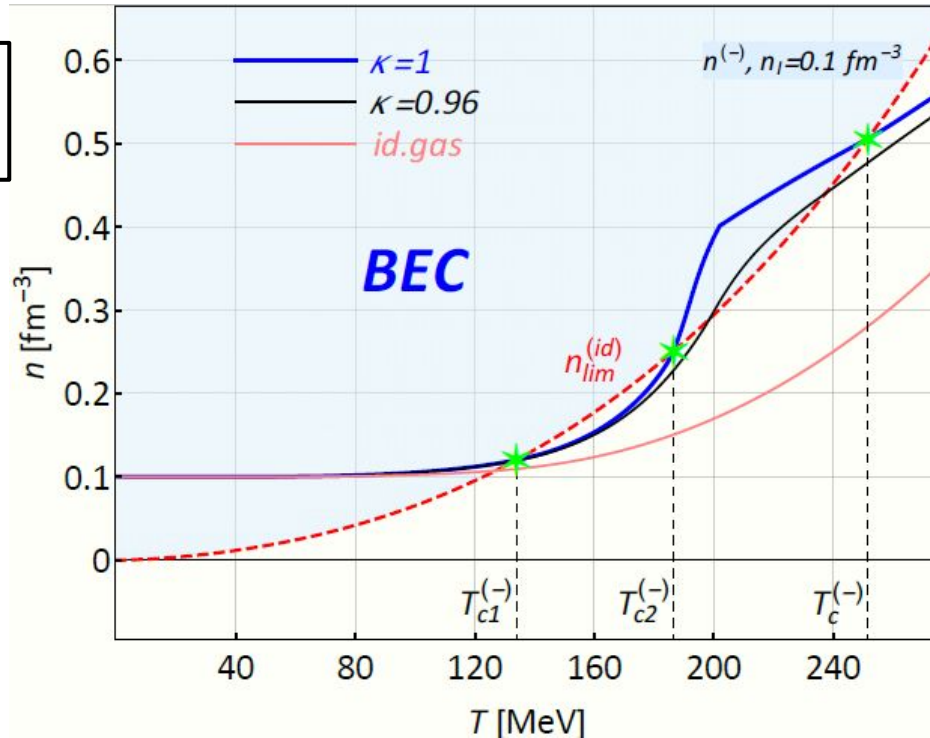
Figure 3. The particle density $n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas.

Type 2 phase transition (2nd order ph. tr.)
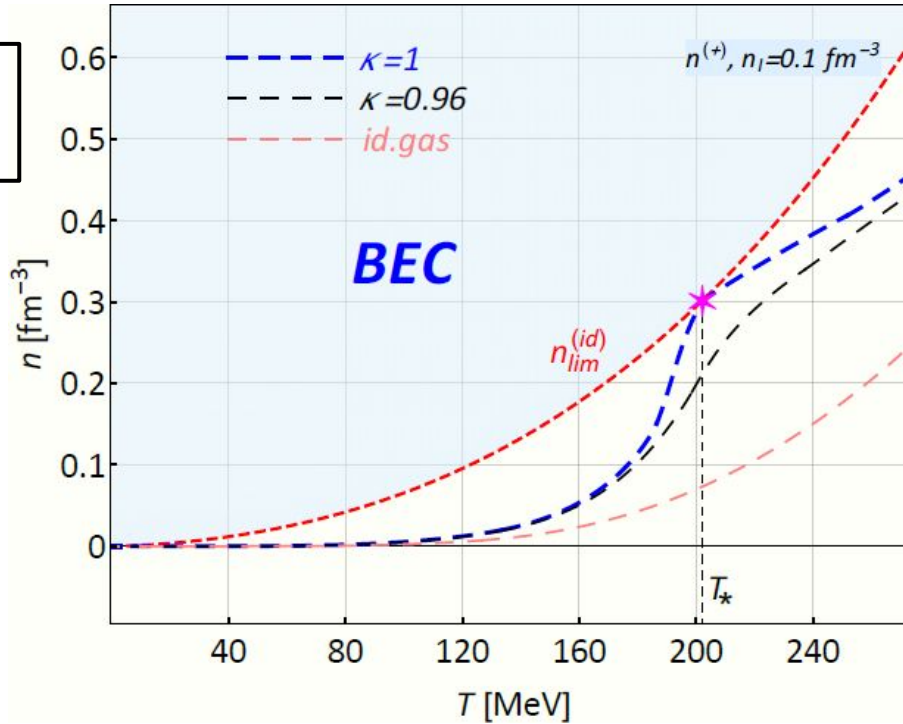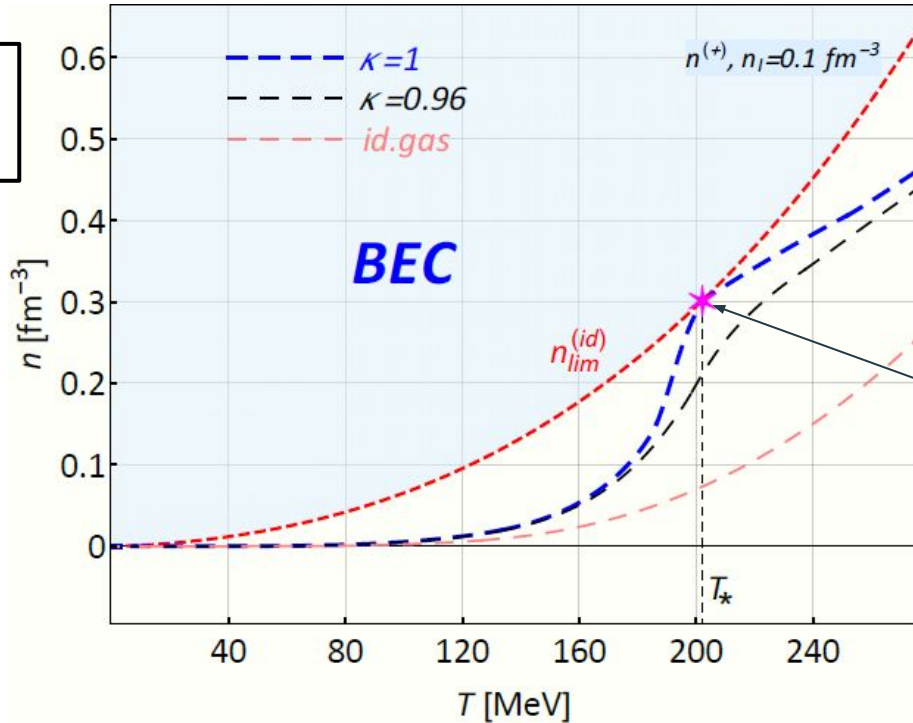
Only particles π⁻ are depicted

2nd order phase transitions

Figure 3. The particle density n$^{(-)}$ versus temperature for the interacting π⁺-π⁻ pion gas.

Type 3 phase transition (2nd order ph. tr.)

Only antiparticles π^+ are depicted

Figure 4. The particle density $n^{(+)}$ versus temperature for the interacting π^+-π^- pion gas.

Type 3 phase transition (2nd order ph. tr.)

Only antiparticles π^+ are depicted

2nd order point-like phase transition

Figure 4. The particle density $n^{(+)}$ versus temperature for the interacting π^+-π^- pion gas.
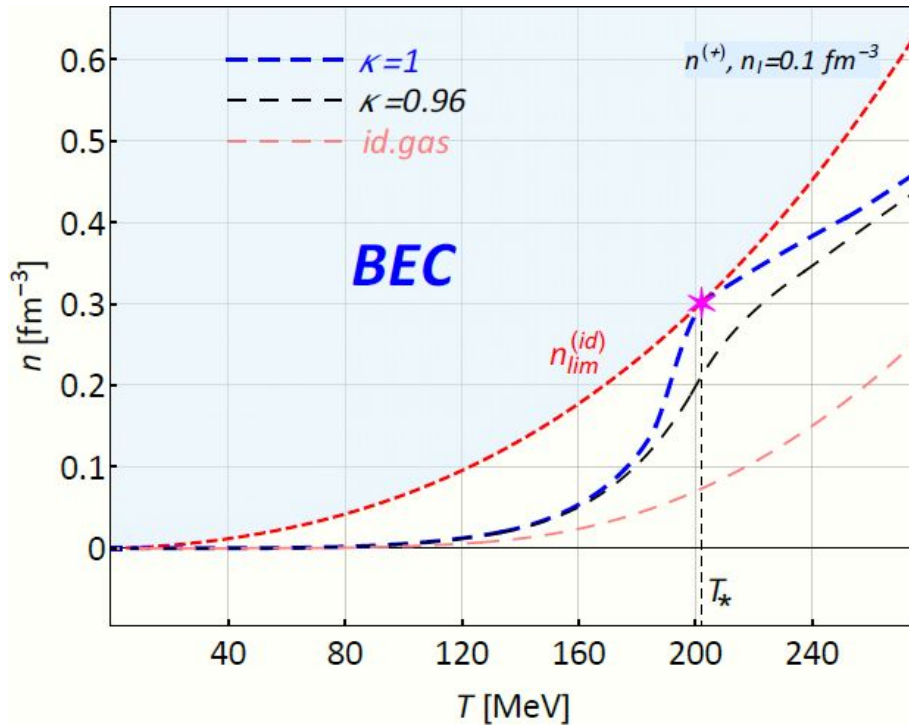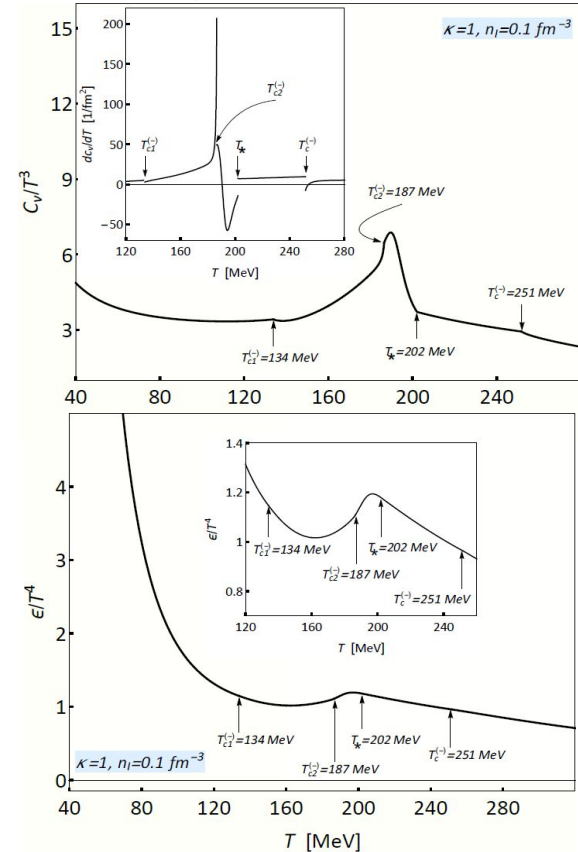
Type 3 phase transition (2nd order ph. tr.)

Figure 4. The particle density $n^{(+)}$ versus temperature for the interacting π^{+}-π^{-} pion gas.

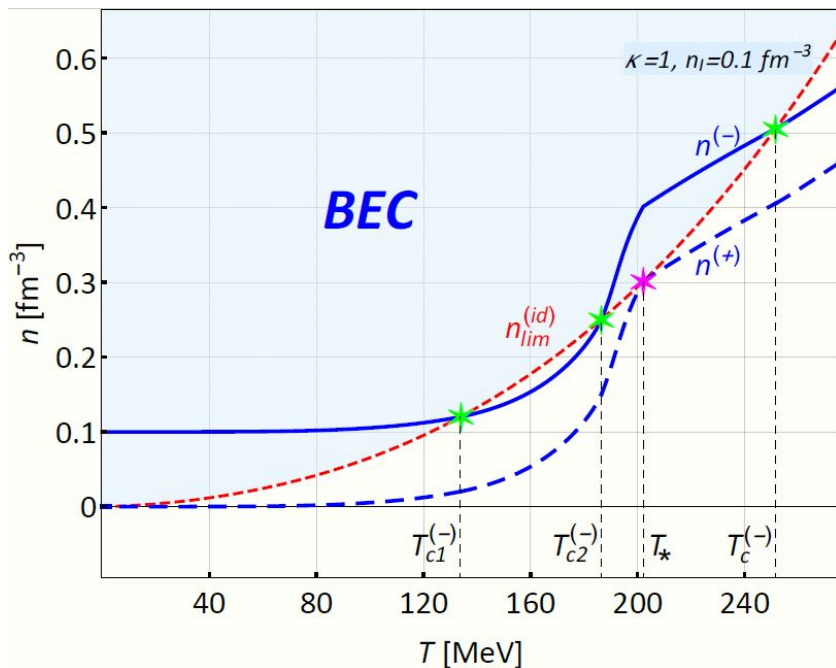# Type 2,3 phase transition (2nd order ph. tr.)[3]

Figure 5. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas. The attraction parameter is **κ = 1**.
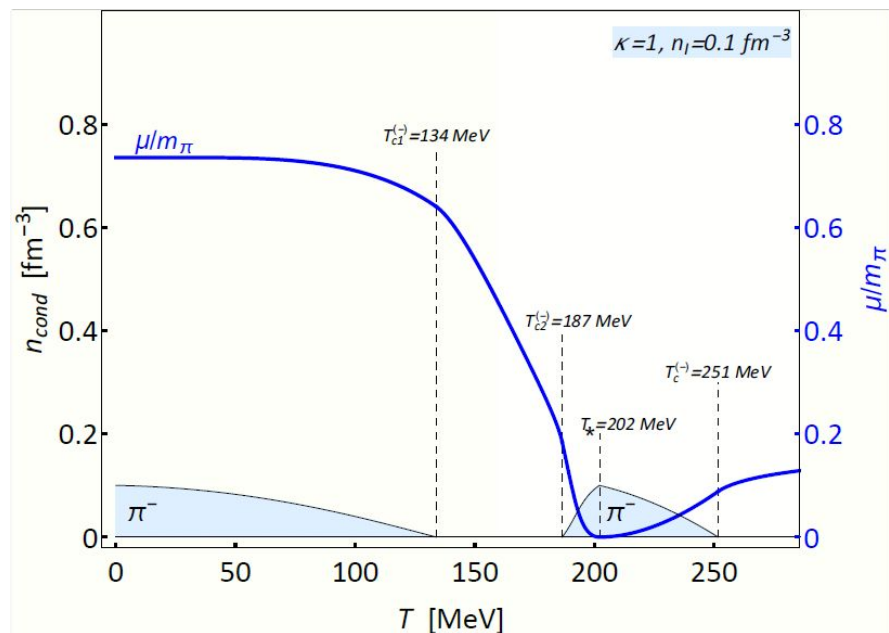
Figure 6. The condensate particle-number densities n_{cond} versus temperature. The attraction parameter is κ = 1.

[3] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Universe 2023, 9, 411 (2023).

What if κ is in over-critical regime?

$$\kappa > 1$$

$$A_c = 396 \text{ MeV} \cdot \text{fm}^3 \,, \qquad n = n_{\text{cond}} + \frac{g}{2\pi^2} \int_0^\infty dk\, k^2 \left\{ \exp\left[\frac{\sqrt{m^2 + k^2} + U(n)}{T} \right] - 1 \right\}^{-1}$$
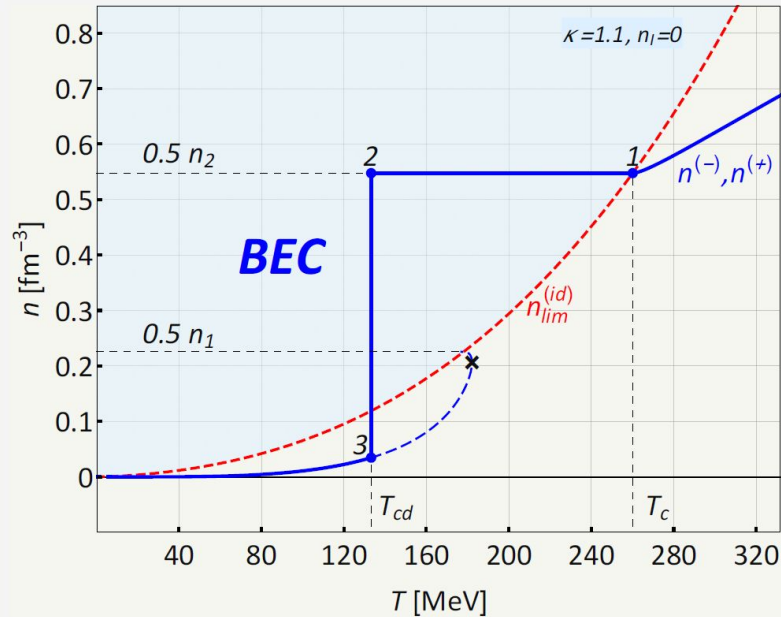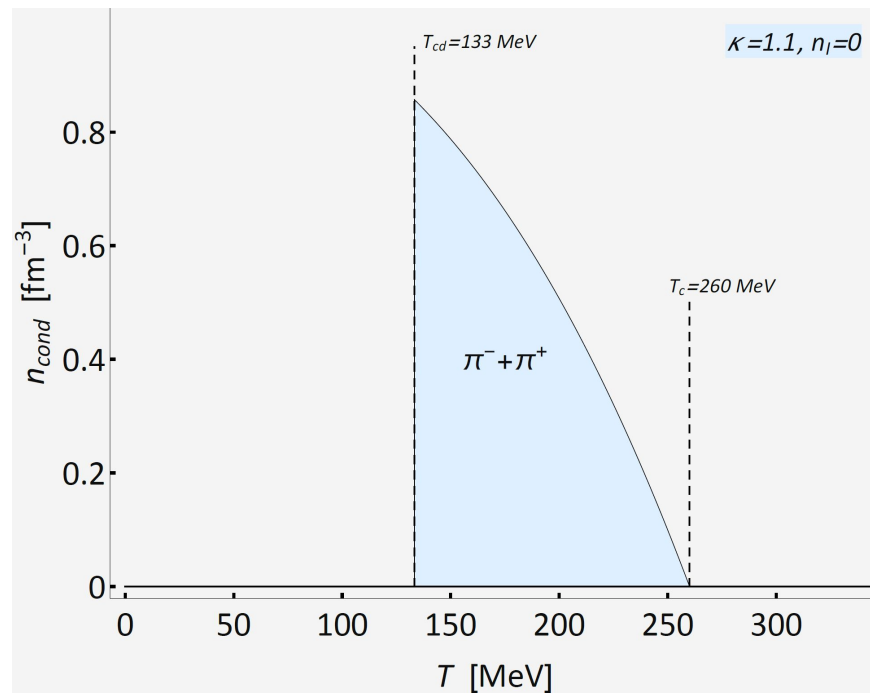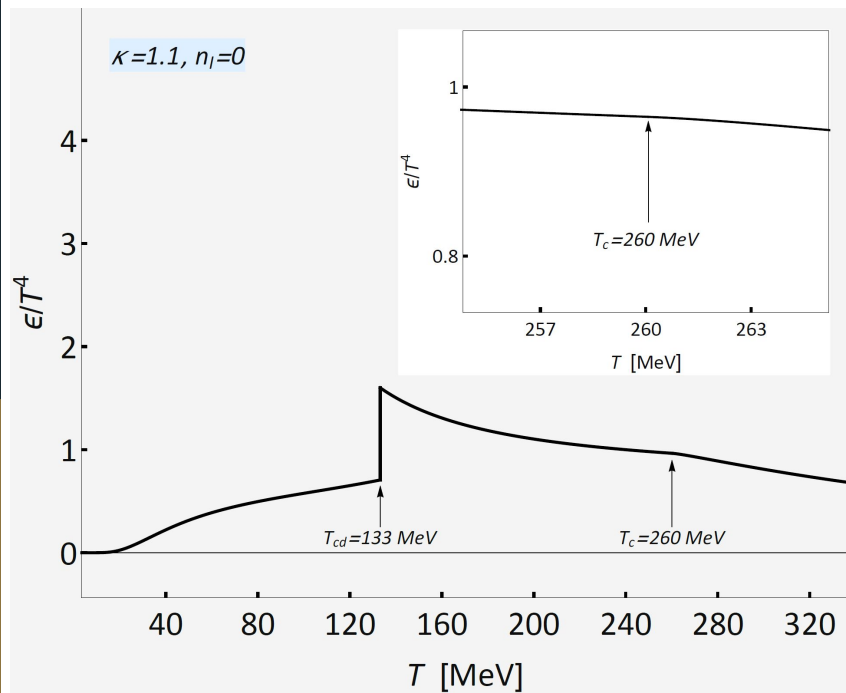

Рис.: Particle density versus temperature for $\kappa = 1.1$, $n_I = 0$.

Left panel: ϵ/T^4 versus T [MeV] with $\kappa=1.1$, $n_I=0$. Inset shows ϵ/T^4 near $T_c=260\ MeV$. Markers at $T_{cd}=133\ MeV$ and $T_c=260\ MeV$. Right panel: n_{cond} [fm^{-3}] versus T [MeV] with $\kappa=1.1$, $n_I=0$, showing the $\pi^-+\pi^+$ region between $T_{cd}=133\ MeV$ and $T_c=260\ MeV$.

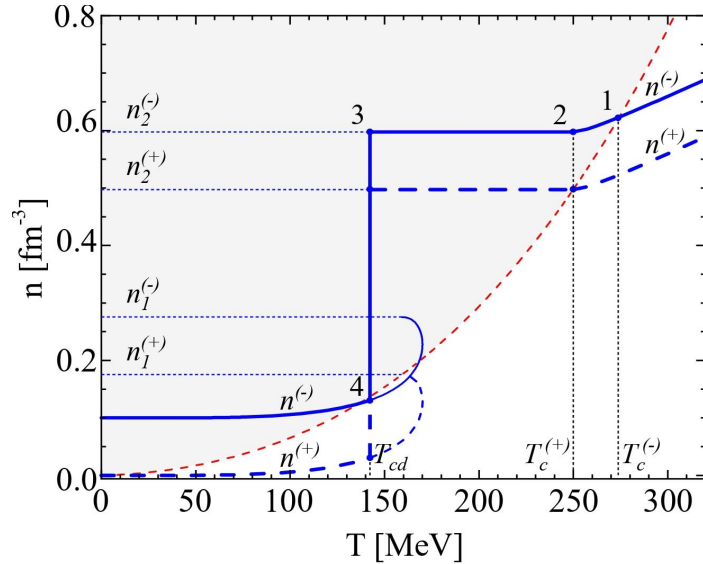# Type 4 phase transition (1st order ph. tr.)[3,4]

Figure 5. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas. The attraction parameter is **κ = 1.1**.

[3] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Universe 2023, 9, 411 (2023).
[4] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Ukr. J. Phys. 2024. Vol. 69, No. 1 (2024).

Type 4 phase transition (1st order ph. tr.)[3,4]

Figure 5. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas. The attraction parameter is **κ = 1.1**.

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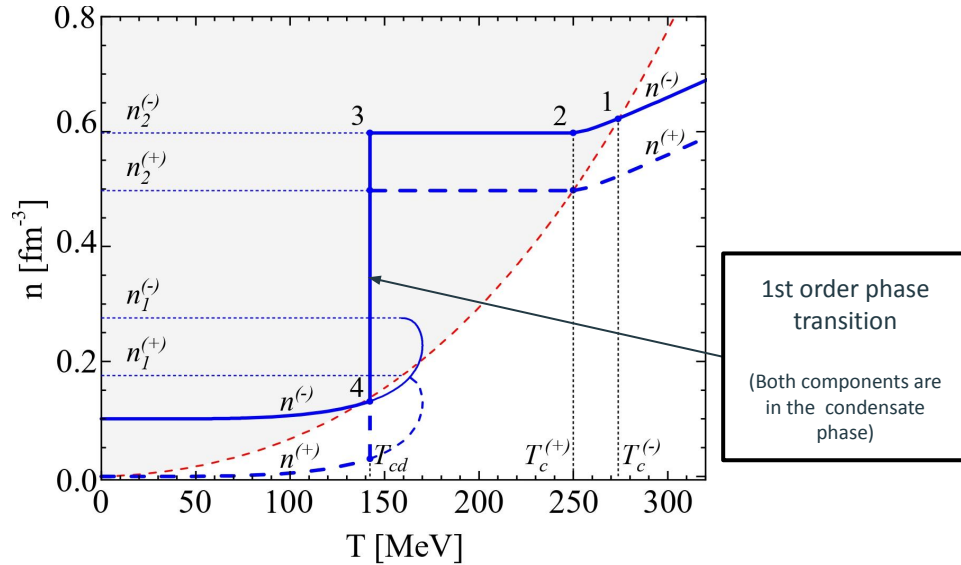
Type 4 phase transition (1st order ph. tr.)[3,4]

Figure 5. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas. The attraction parameter is **$\kappa = 1.1$**.

$$n_1 = \sqrt{\frac{m}{B}}\left(\kappa - \sqrt{\kappa^2 - 1}\right), \quad n_2 = \sqrt{\frac{m}{B}}\left(\kappa + \sqrt{\kappa^2 - 1}\right),$$

[3] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Universe 2023, 9, 411 (2023).
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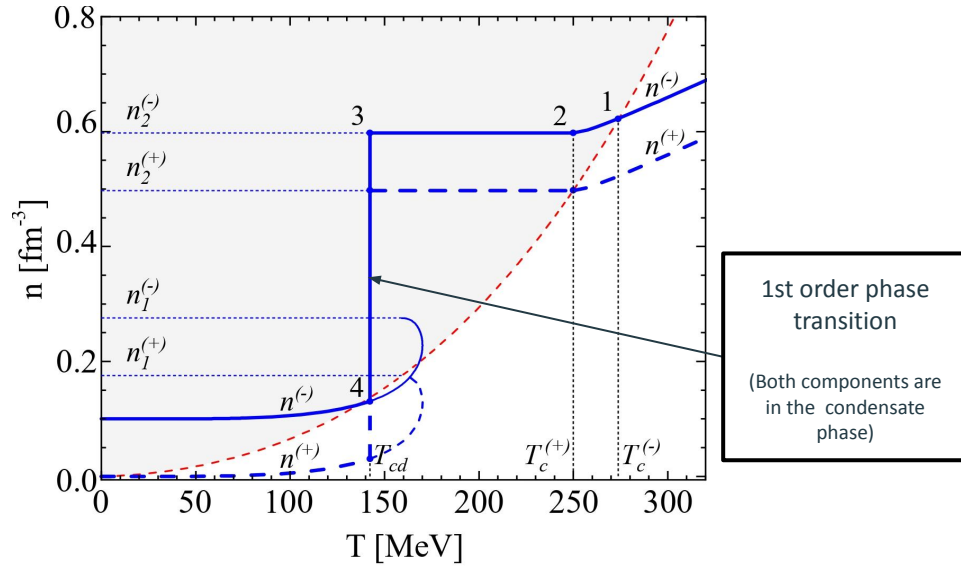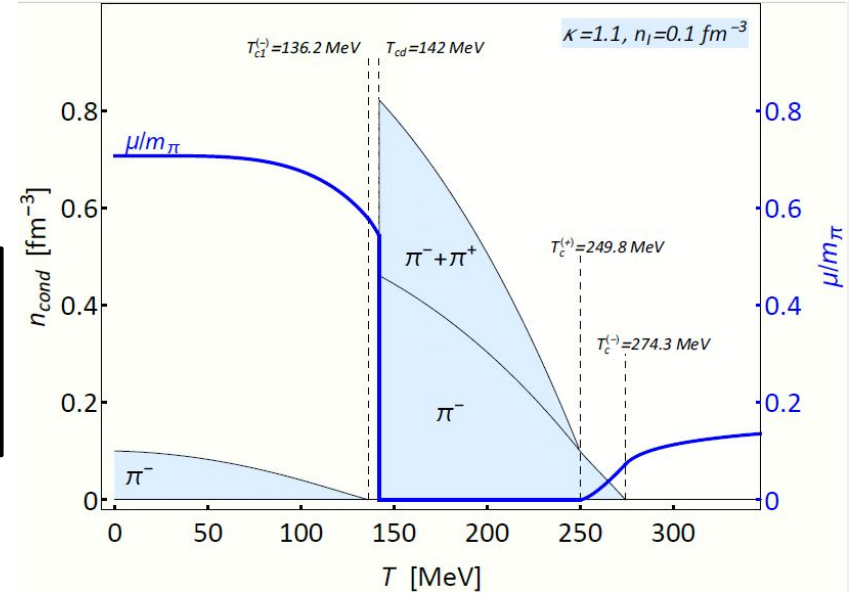
Type 4 phase transition (1st order ph. tr.)[3,4]

Figure 5. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the interacting π^+-π^- pion gas. The attraction parameter is **κ = 1.1**.

1st order phase transition

(Both components are in the condensate phase)

$$n_1 = \sqrt{\frac{m}{B}}\left(\kappa - \sqrt{\kappa^2 - 1}\right), \quad n_2 = \sqrt{\frac{m}{B}}\left(\kappa + \sqrt{\kappa^2 - 1}\right),$$

[3] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, V. Karpenko, I. Mishustin, H. Stoecker, Universe 2023, 9, 411 (2023).
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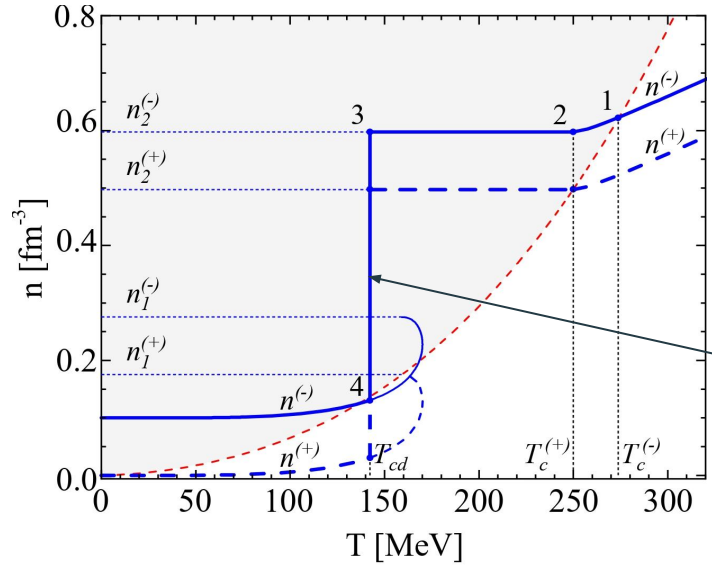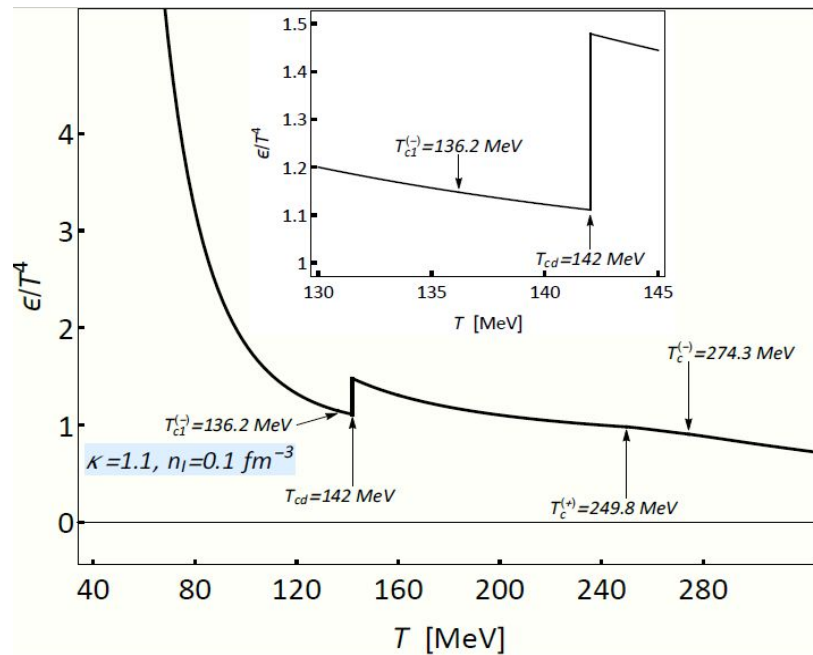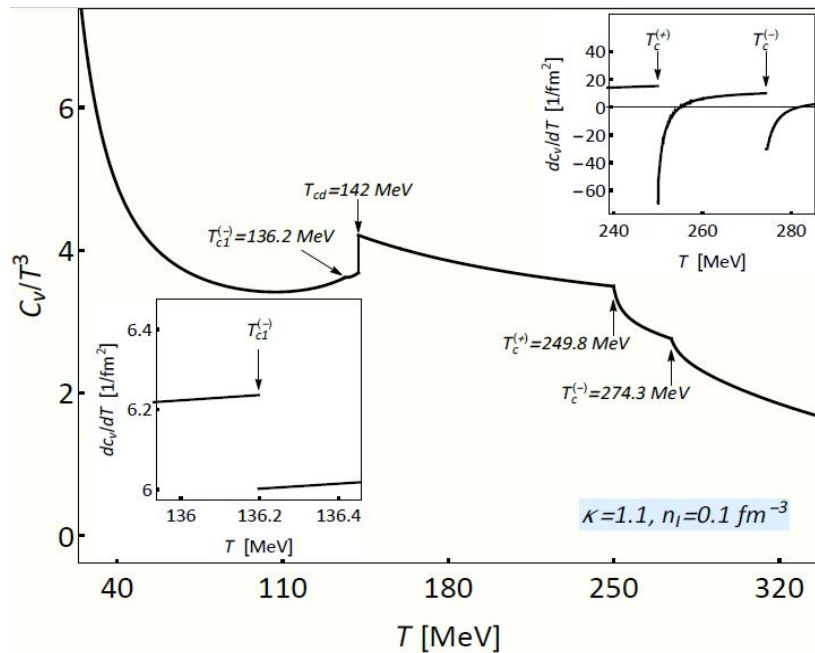# Type 4 phase transition (1st order ph. tr.)[3,4]
Proof

Conclusions

❏ The scalar and thermodynamic mean-field models were used to describe the 2-component system of interacting mesons (bosons) at high temperatures in the presence of condensate.

❏ In the case of the conservation of isospin, 4 types of phase transitions are defined in a relativistic bosonic system with repulsion and attraction: **type 1** - the one-time transition of the dominant component (2nd order), **type 2** - the multiple phase transition of the dominant component (2nd order), **type 3** - the point (virtual) phase transition of the second component (2nd order), **type 4** - the transition of both components into the condensate phase (1st order).

❏ When both components are in the condensate phase, the thermal particles have no additional charge, that is, the isospin (charge) density of the thermal particles is zero. Additional charge accumulates only in condensate particles.

The Grand Canonical Ensemble is not suitable for describing the bosonic system in the condensate phase

Ideal nonrelativistic 1-component gas
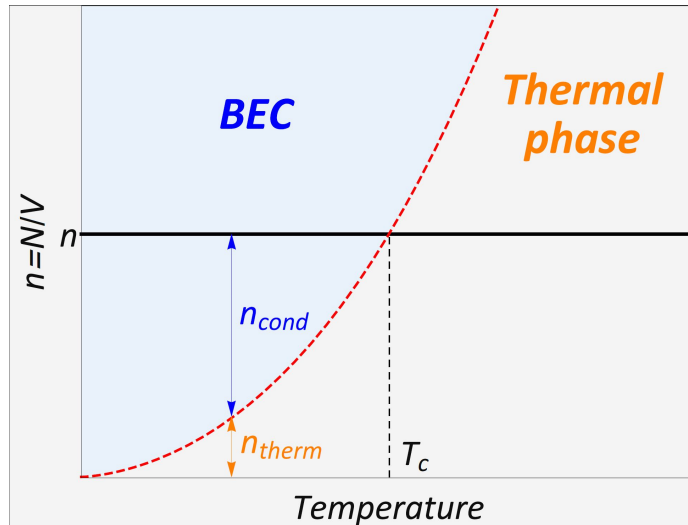
$$\frac{N}{V} = n_{\text{cond}} + \int \frac{d^3 k}{(2\pi\hbar)^3} \frac{1}{\exp\left[\frac{k^2/2m - \mu}{T}\right] - 1}\Big|_{\mu=0}$$

Condition for the condensate formation:

$$\mu_I = 0$$

In the interval $0 < T < T_c$
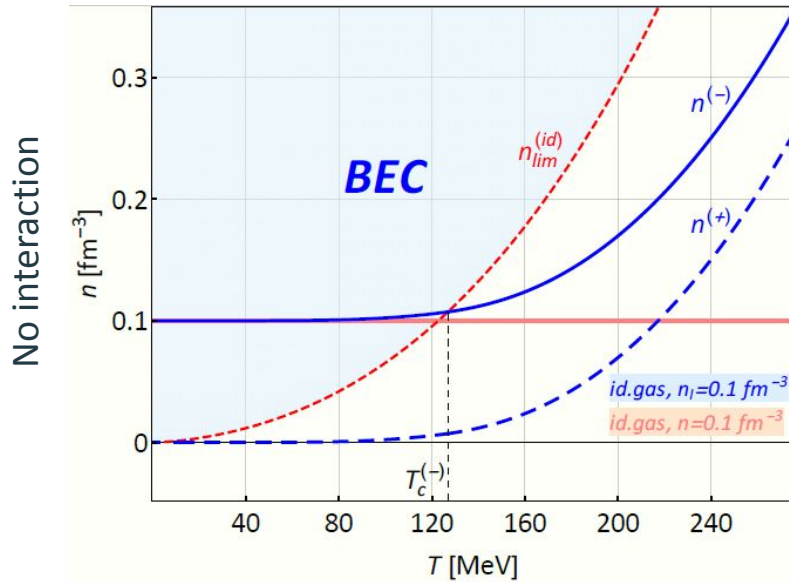
Ideal 2-component relativistic gas

Figure 1. The particle-number densities $n^{(+)}$, $n^{(-)}$ versus temperature for the ideal π^+-π^- pion gas. The total isospin density is kept constant, $n_I = 0.1$ fm^{-3}.

No interaction

$$n_I = \int \frac{d^3 k}{(2\pi)^3} \left[f_{\text{BE}}(\omega_k, \mu_I) - f_{\text{BE}}(\omega_k, -\mu_I) \right]$$

$$\omega_k = \sqrt{m^2 + \mathbf{k}^2}$$

Both components cannot be in the condensate phase at the same time:

$$m - \mu_I = 0,$$
$$m + \mu_I = 0$$

$$\Longrightarrow \quad m = 0$$
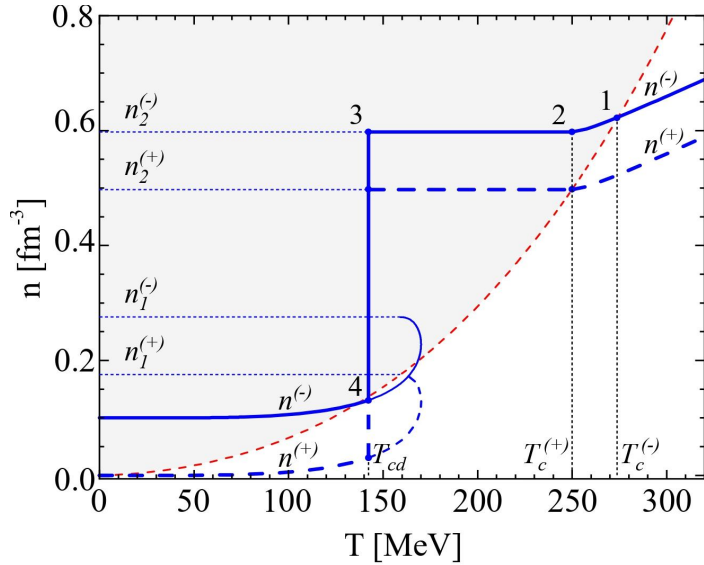$$\mu_I = 0$$

Condition for the condensate formation:

$$\mu_I = m \qquad \text{In the interval} \quad 0 < T < T_c$$

In the condensate phase the chemical potential is not a free parameter

The thermodynamic mean-field model

$$m + U(n) - \mu_I = 0$$

$$m + U(n) + \mu_I = 0$$

$$\Longrightarrow$$

$$\mu_I = 0$$

$$m + U(n) = 0$$

The scalar field model

$$M(\sigma) - \mu_I = 0$$

$$M(\sigma) + \mu_I = 0$$

$$\Longrightarrow$$

$$\mu_I = 0$$

$$M(\sigma) = 0$$

GCE: canonical variables are (T, μ_I)

CE: canonical variables are (T, n_I)

Conclusion

-     We have demonstrated that the Grand Canonical Ensemble is *not suitable* for describing a multi-component bosonic system in the presence of condensate phase. In particular, it cannot describe the condensate state in the system of particles and antiparticles. The reason is that the chemical potential is not a free parameter in the condensate phase, its values are determined by the necessary condition for condensate formation. As we have shown, these statements are valid in interacting bosonic systems as well as in an ideal bosonic gas.

Thank you for attention!

Pressure $P = T \ln Z / V$

$$p = p_{\mathrm{kin}}(M, T) + P_{\mathrm{ex}}(\sigma), \quad p_{\mathrm{kin}}(M, T) = \frac{2}{3} \int \frac{d^3 k}{(2\pi)^3} \frac{k^2}{\omega_{\boldsymbol{k}}} f_{\mathrm{BE}} \left[\omega_k(\sigma), \mu_I = 0 \right]$$

Energy density, entropy density $\quad s = (\varepsilon + P)/T$

$$\varepsilon = \varepsilon_{\mathrm{cond}} + 2 \int \frac{d^3 k}{(2\pi)^3} \omega_{\boldsymbol{k}} f_{\mathrm{BE}} \left[\omega_k(\sigma), \mu_I = 0 \right] - P_{\mathrm{ex}}(\sigma)$$

$$s = s_{\mathrm{cond}} + \frac{2}{T} \int \frac{d^3 k}{(2\pi)^3} \left(\omega_{\boldsymbol{k}} + \frac{k^2}{3\omega_{\boldsymbol{k}}} \right) f_{\mathrm{BE}} \left[\omega_k(\sigma), \mu_I = 0 \right]$$